ON ROBIN'S INEQUALITY

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Abstract

Robin conjectured that $\sigma(n) < e^{\gamma} n \log\log n$ holds for all $n > 7!$, where $\sigma(n)$ is the sum of the divisors of n and γ denotes the Euler constant. Robin showed that the validity of this inequality is equivalent to the Riemann Hypothesis. Here we show that the set of n's failing this inequality has a very small counting function: the number of such $n \le x$ is $O(x^{\varepsilon})$ for any $\varepsilon > 0$ and $x > x(\varepsilon)$.

1. Introduction

Robin conjectured that $\sigma(n) < e^{\gamma} n \log\log n$ holds for all positive integers $n > 7!$, where $\sigma(n)$ is the sum of divisors and γ is the Euler constant. Ramanujan [7] proved that the Riemann Hypothesis implies the above inequality and Robin proved that the above inequality is equivalent to the Riemann Hypothesis. No numerical counterexample (larger than 7!) is known. Several papers looked at infinite classes of integers n for which this could be proved to hold such as

(i) odd and greater than 9 [4];

(ii) square-free and greater than 30 [4];

(iii) a sum of two squares greater than 720 [2];

(iv) not divisible by the fifth power of a prime [4];

(v) not divisible by the seventh power of a prime [8];

(vi) not divisible by the eleventh power of a prime [3];

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(vii) not divisible by the twentieth power of a prime [6];

(viii) not divisible by the twenty-first power of a prime [1].

All sets listed at (i)–(viii) are of density smaller than 1 as subsets of integers (and (iii) is of density 0). So, one might ask whether it can be shown that the set of n's failing Robin's inequality (a finite set under the Riemann Hypothesis) is at least of asymptotic density 0. It turns out that this has been done in [10]. No explicit bounds for the counting function of the exceptional n were given in [10]. This is the primary purpose of our paper. Let

$$\mathcal{N}R(x) = \{7! < n \le x : \sigma(n) \ge e^{\gamma} n \log\log n\}.$$

In this paper we give upper bounds on $\#\mathcal{N}R(x)$. Numbers n for which $\sigma(n)$ is very large (so they might have a chance of belonging to $\mathcal{N}R$ in the absence of the Riemann hypothesis) are the highly composite numbers and the colossally abundant numbers first studied by Ramanujan [7] and later by Erdős and Nicolas [5]. Throughout the paper, we use Landau's symbols O and o as well as Vinogradov's symbols $\ll$ and $\gg$ with their regular meanings. Recall that if $f,\ g$ are functions of a real parameter x then $f(x) = O(g(x)),\ f(x) \ll g(x)$ and $g(x) \gg f(x)$ are all equivalent to the fact that the inequality $|f(x)| < Kg(x)$ holds with some positive constant K for all $x > x_0$, whereas $f(x) = o(g(x))$ means that $f(x)/g(x)$ tends to 0 as x tends to infinity.

2. Theorems and Proofs

Theorem 1. *For $x \ge 3$, we have*

$$\#\mathcal{N}R(x) = O\left(\frac{x}{\log\log x}\right).$$

Proof. Recall that

$$\sum_{n \le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x + O(\log x).$$

Let $n \in \mathcal{N}R(x) \cap [x/2, x]$. Then $\sigma(n)/n \ge e^{\gamma} \log\log n \ge e^{\gamma} \log\log(x/2)$. We thus get

$$e^{\gamma} \log\log(x/2) \# \left(\mathcal{N}R(x) \cap [x/2, x]\right) \le \sum_{x/2 \le n \le x} \frac{\sigma(n)}{n} \ll x,$$

therefore

$$\# \left(\mathcal{N}R(x) \cap [x/2, x]\right) \ll \frac{x}{\log\log x}.$$

Replacing now x by $x/2$, then by $x/4$, etc. and summing up the resulting estimates we get the desired conclusion. □

The next result presents a better upper bound on $\#\mathcal{N}R(x)$.

Theorem 2. *For $x \geq 3$, we have*

$$\#\mathcal{N}R(x) = O\left(\frac{x}{\sqrt{\log x}}\right).$$

Proof. Let $\tau(n)$ be the number of divisors of n. Recall that

$$\sum_{n \leq x} \tau(n) = x \log x + O(x).$$

Let $\mathcal{N}_1(x)$ be the set of $n \in [x/2, x]$ such that $\tau(n) \geq (\log x)^{1.5}$. The above estimate gives us that

$$(\log x)^{1.5} \#\mathcal{N}_1(x) \leq \sum_{x/2 \leq n \leq x} \tau(n) = O(x \log x),$$

which gives

$$\#\mathcal{N}_1(x) = O\left(\frac{x}{\sqrt{\log x}}\right).$$

We next show that for $x > x_0$, $\mathcal{N}R(x) \cap [x/2, x] \subset \mathcal{N}_1(x)$. Indeed, let $n \in [x/2, x]$ not in $\mathcal{N}_1(x)$. Then $\tau(n) < (\log x)^{1.5}$. Next,

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} = \sum_{\substack{d|n \\ d < (\log x)^{1.5}}} \frac{1}{d} + \sum_{\substack{d|n \\ d \geq (\log x)^{1.5}}} \frac{1}{d} =: S_1(n) + S_2(n).$$

Clearly,

$$S_2(n) \leq \frac{\tau(n)}{(\log x)^{1.5}} < 1 \qquad \text{since} \qquad n \notin \mathcal{N}_1(x).$$

As for $S_1(n)$, we can extend it over all positive integers $d < (\log x)^{1.5}$ getting that

$$S_1(n) \leq \sum_{d < (\log x)^{1.5}} \frac{1}{d} < 1 + \log((\log x)^{1.5}) = 1 + 1.5 \log\log x.$$

Hence, for $n \notin \mathcal{N}_1(x)$, we have that

$$\frac{\sigma(n)}{n} = S_1(n) + S_2(n) < 2 + 1.5 \log\log x.$$

The right–hand side above is smaller than $1.78 \log\log(x/2) < e^{\gamma} \log\log n$ when $n \in [x/2, x]$ and $x > 10^{600}$. This shows that

$$\#\left(\mathcal{N}R(x) \cap [x/2, x]\right) \leq \#\mathcal{N}_1(x) = O\left(\frac{x}{\sqrt{\log x}}\right) \qquad \text{for} \qquad x > 10^{600}.$$

Replacing now x by $x/2$, then by $x/4$, etc. and summing up the resulting inequalities we get the desired conclusion. ◻

Remark 1. The exponent 1/2 on the logarithm in the upper bound for $\#\mathcal{N}R(x)$ can be improved to $e^{\gamma}-1-\varepsilon$ for any $\varepsilon>0$ and $x>x(\varepsilon)$.

While the previous two theorems give some answer to the motivating question of our note, namely finding explicit upper bounds for the counting function of the numbers failing Robin's inequality, they are a bit disappointing in that they do not seem to be related in any way to the Riemann Hypothesis, which in general appears when working with estimates concerning $\pi(x)$, $\theta(x)$ or $\psi(x)$. Our last result points out this connection.

Theorem 3. *The estimate*

$$\#\mathcal{N}R(x)=O(x^{\varepsilon})$$

holds for any $\varepsilon>0$ *and* $x>x(\varepsilon)$.

Proof. As usual, we take $n\in\mathcal{N}R(x)\cap[x/2,x]$. Then

$$\frac{\sigma(n)}{n}\geq e^{\gamma}\log\log n\geq e^{\gamma}\log\log(x/2)=e^{\gamma}\log\log x\left(1+O\left(\frac{1}{\log x}\right)\right). \tag{1}$$

We have

$$\begin{aligned}\frac{\sigma(n)}{n} &\leq \frac{n}{\phi(n)}=\prod_{p\mid n}\left(1-\frac{1}{p}\right)^{-1}\\ &= \prod_{p\leq\log x}\left(1-\frac{1}{p}\right)^{-1}\prod_{\substack{p\leq\log x\\ p\nmid n}}\left(1-\frac{1}{p}\right)\prod_{\substack{p>\log x\\ p\mid n}}\left(1-\frac{1}{p}\right)^{-1}\\ &=: P_1(x)^{-1}P_2(n,x)P_3(n,x)^{-1}. \end{aligned}\tag{2}$$

We take logarithms in (1) and use (2) to get

$$\log P_1(x)^{-1}+\log P_2(n,x)+\log P_3(n,x)^{-1}\geq\log(e^{\gamma}\log\log x)+O\left(\frac{1}{\log x}\right). \tag{3}$$

By a result of Vinogradov [9]

$$\prod_{p\leq\log x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log\log x}\Big(1+O(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5}))\Big)$$

for some constant $a>0$. Hence,

$$\log P_1(x)^{-1}=\log(e^{\gamma}\log\log x)+O(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})). \tag{4}$$

Further, $\omega(n) < 2\log n/\log\log n$ for large x. Since the interval $(\log x, 4\log x)$ contains $(3+o(1))\log x/\log\log x$ primes as $x \to \infty$, it follows that

$$\begin{aligned}
\log P_3(n,x)^{-1} &= O\left(\sum_{\substack{p>\log x\\ p\mid n}} \frac{1}{p}\right) = O\left(\sum_{\log x<p<4\log x} \frac{1}{p}\right)\\
&= O(\log\log(4x) - \log\log x + \exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})\\
&= O\left(\log\left(1+\frac{\log 4}{\log x}\right) + \exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})\right)\\
&= O\left(\frac{1}{\log x} + \exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})\right)\\
&= O\left(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})\right). \qquad (5)
\end{aligned}$$

Putting (4) and (5) into (3), we get that

$$\begin{aligned}
-\log P_2(n,x) &\leq O\left(\frac{1}{\log x}\right) + O(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5}))\\
&= O\left(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})\right). \qquad (6)
\end{aligned}$$

However,

$$-\log P_2(n,x) \gg \sum_{\substack{p\leq \log x\\ p\nmid n}} \frac{1}{p}.$$

Let m be the number of primes involved in $P_2(n,x)$. We then get that the right–hand side above is at least $m/\log x$, which together with (6) gives

$$m \ll \frac{\log x}{\exp(a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})} \ll \frac{\log x}{\exp((\log\log x)^{1/2})}. \qquad (7)$$

The number of ways of choosing m primes among the first $\pi(\log x)$ is

$$\begin{aligned}
\binom{\pi(\log x)}{m} &\ll \frac{\pi(\log x)^m}{m!} \ll e^m m^{1/2}\left(\frac{\pi(\log x)}{m}\right)^m\\
&= x^{o(1)} O\left(\frac{\exp((\log\log x)^{1/2})}{\log\log x}\right)^{O\left(\frac{\log x}{\exp((\log\log x)^{1/2})}\right)}\\
&= x^{o(1)}\exp\left(O\left(\frac{\log x(\log\log x)^{1/2}}{\exp((\log\log x)^{1/2})}\right)\right) = x^{o(1)}.
\end{aligned}$$

In the above, we used the Stirling formula to approximate the factorial. So, there are $x^{o(1)}$ subsets of primes $P \subset [1, \log x]$ which can be the sets of missing primes out of n up to $\log x$. For each one of these,

$$\prod_{p\in P} p \le (\log x)^m = \exp\left(O\left(\frac{\log x \log\log x}{\exp((\log\log x)^{1/2})}\right)\right) = x^{o(1)}.$$

Thus, letting Q_P be such that

$$Q_P := \prod_{\substack{p\le \log x\\ p\notin P}} p,$$

we have that $Q_P \mid n$. By the Prime Number Theorem,

$$\begin{aligned} Q_P &= \left(\prod_{p\in P} p\right)^{-1} \prod_{p\le \log x} p = x^{o(1)} \exp\left(\sum_{p\le \log x} \log p\right)\\ &= x^{o(1)} \exp(\log x + O(\exp(-a(\log\log x)^{3/5}(\log\log\log x)^{-1/5})))\\ &= x^{1+o(1)}. \end{aligned}$$

Since $Q_P \mid n$, it follows that $n = Q_P n_1$. Since $n \le x$ and $Q_P = x^{1+o(1)}$, it follows that $n_1 = x^{o(1)}$. To recap, if $n \in \mathcal{N}R(x) \cap [x/2, x]$ then there is a positive integer m satisfying (7) and a set of primes $P \subset [1, \log x]$ of cardinality exactly m such that n is divisible by Q_P, the product of all primes $p \le \log x$, not in P. The number m and set P can be chosen in $x^{o(1)}$ ways. Once P is chosen, Q_P is uniquely determined of size $x^{1+o(1)}$, so $n = Q_P n_1$, where n_1 can be chosen in $x^{o(1)}$ ways as well. Putting all together, we get at most $x^{o(1)}$ choices for n. Now we replace x by $x/2$, then $x/4$, etc. and sum the resulting estimates. □

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