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ASCENT SEQUENCES AND WEAK ASCENT SEQUENCES AVOIDING A QUADRUPLE OF LENGTH-3 PATTERNS

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Abstract

We say that two sets of patterns B and C are A-Wilf-equivalent if the number of ascent sequences of length n that avoid all the patterns in B equals the number of ascent sequences of length n that avoid all the patterns in C, for all $n \geq 0$. Similarly, *WA*-Wilf-equivalence refers to weak ascent sequences. Here, we show that the number of A-Wilf-equivalence classes among quadruples of length-3 patterns is 74 and the number of *WA*-Wilf-equivalence classes among quadruples of length-3 patterns is either 228 or 229. The main tool is generating trees; bijective methods are also sometimes used.

1. Introduction

An *ascent*, short for *ascent index*, in an integer sequence $s_1 s_2 \cdots s_m$ is an index $1 \leq j \leq m-1$ such that $s_j < s_{j+1}$. An *ascent sequence* $a_1 a_2 \cdots a_n$ is a sequence of non-negative integers that satisfies $a_1 = 0$ and $a_i \leq \text{asc}(a_1 a_2 \cdots a_{i-1}) + 1$ for $1 < i \leq n$, where $\text{asc}(a_1 a_2 \cdots a_k)$ is the number of ascents in the sequence $a_1 a_2 \cdots a_k$. For example, the sequence 0102310401 is an ascent sequence, whereas 010300 is not. Bousquet-Mèlou, Claesson, Dukes, and Kitaev [4] linked ascent sequences to (2+2)-free posets. Since that discovery, ascent sequences have been explored in numerous papers, revealing connections to various other combinatorial structures (see, for instance, [9–12, 14–16] and [13, Section 3.2.2]).

Let $s = s_1 s_2 \cdots s_n$ be any sequence of nonnegative integers and $\tau = \tau_1 \cdots \tau_m$ be any *pattern*, that is, a word in $\{0, \ldots, \ell\}^m$ which contains each letter $0, 1, \ldots, \ell$ for some $m \geq 1$ and $\ell \geq 0$. The reduced form of s is obtained by replacing each

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occurrence of the smallest entry in s with 0, each occurrence of the next smallest entry with 1, and so on. Thus the reduced form of 24542744 is 01210311, and a reduced form is always a pattern. We say the sequence s *contains* the pattern τ if s has a subsequence that is order isomorphic to τ, that is, there is a subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_m}$ such that its reduced form is τ. Otherwise, s is said to *avoid* τ. For instance, the ascent sequence 01013043351 has two occurrences of the pattern 110, namely, the subsequences 110 and 331 whose reduced forms are both 110, but avoids the pattern 3120. We denote the set of all ascent sequences that avoid a list of patterns $\tau^{(1)}, \ldots, \tau^{(s)}$ by $A_n(\tau^{(1)}, \ldots, \tau^{(s)})$ or $A_n(\{\tau^{(1)}, \ldots, \tau^{(s)}\})$. We say that two sets of patterns P and Q are A-Wilf-equivalent, denoted $P \overset{a}{\sim} Q$, if $|A_n(P)| = |A_n(Q)|$ for every n.

There are 13 patterns of length 3: 000, 001, 010, 100, 011, 101, 110, 012, 021, 102, 120, 201, and 210. The number of A-Wilf-equivalence classes among single patterns of length 3 is 9 [12]. The number of A-Wilf-equivalence classes among pairs of patterns of length 3 is 35 [1]. The number of A-Wilf-equivalence classes among triples of length-3 patterns is 62 (see [7]). Here is the first of our two main results.

Theorem 1. *The number of A-Wilf-equivalence classes among quadruples of length-3 patterns is 74.*

The concept of ascent sequence has been extended to weak ascent sequences. Bényi, Claesson, and Dukes [2] described a connection between weak ascent sequences and permutations avoiding a bivincular pattern of length four, restricted upper-triangular binary matrices, and factorial posets that are weakly (3+1)-free. A *weak ascent sequence* is a sequence $a = a_1 a_2 \cdots a_n$ of nonnegative integers such that $a_1 = 0$ and $a_i \leq 1 + \mathrm{wasc}(a_1 a_2 \cdots a_{i-1})$ for $i = 2, \ldots, n$, where $\mathrm{wasc}(a_1 a_2 \cdots a_m)$ is the number of *weak ascents* in the sequence $a_1 a_2 \cdots a_m$, defined as the number of positions j such that $a_j \leq a_{j+1}$. The set of all weak ascent sequences of length n is denoted by $W\!A_n$. For example, the weak ascent sequence $00203413 \in W\!A_8$ contains the pattern 011 and avoids the pattern 321. We denote the set of weak ascent sequences of length n that avoid π by $W\!A_n(\pi)$ and similarly for sets of patterns. We say that two sets of patterns P and Q are $W\!A$-Wilf equivalent, denoted $P \overset{w}{\sim} Q$, if $|W\!A_n(P)| = |W\!A_n(Q)|$ for every $n \geq 0$. The $W\!A$-Wilf-equivalence classes for single patterns of length three are considered in [3]. The number WAS_2 of $W\!A$-Wilf-equivalence classes among pairs of patterns of length 3 satisfies $59 \leq \mathrm{WAS}_2 \leq 61$ and the number WAS_3 of $W\!A$-Wilf-equivalence classes among triples of patterns of length 3 satisfies $150 \leq \mathrm{WAS}_3 \leq 155$, see [8]. Here is our second main result.

Theorem 2. *The number $W\!AS_4$ of $W\!A$-Wilf-equivalence classes among quadruples of length-3 patterns is either 228 or 229.*

The slight ambiguity in Theorem 2 is because the status of Class 215 is left open in Table 2 below.

In the last section, we state some results for the number of A-Wilf-equivalence and WA-Wilf-equivalence classes among sets of k length-3 patterns for $k \geq 5$.

2. Generating Trees and the Strategy for the Proofs

Let P be any set of patterns such that the length of each pattern is at least two. Define $A(P) = \cup_{n=0}^{\infty} A_n(P)$ and $WA(P) = \cup_{n=0}^{\infty} WA_n(P)$. We will construct a pattern-avoidance generating tree $\mathcal{T}(P)$ (see [7, 8]) for the class of pattern-avoiding ascent sequences $A(P)$ and a pattern-avoidance generating tree $\mathcal{T}'(P)$ for the class of pattern-avoiding weak ascent sequences $WA(P)$. Starting with the root 0 which stays at level 1, we construct in a recursive manner the non-root nodes of the tree $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$) such that the nth level of the tree consists of exactly the elements of $A_n(P)$ (resp. $WA(P)$) arranged so that the parent of an ascent sequence $a_1 \cdots a_n \in A_n(P)$ (resp. $a_1 \cdots a_n \in WA_n(P)$) is the unique ascent sequence $a_1 \cdots a_{n-1} \in A_{n-1}(P)$ (resp. $a_1 \cdots a_{n-1} \in WA_{n-1}(P)$). The children of $a_1 \cdots a_{n-1} \in A_{n-1}(P)$ (resp. $a_1 \cdots a_{n-1} \in WA_{n-1}(P)$) are obtained from the set $\{a_1 \cdots a_{n-1}a_n \mid a_n = 0, 1, \ldots, \mathrm{asc}(a_1 \cdots a_{n-1}) + 1\}$ (resp. $\{a_1 \cdots a_{n-1}a_n \mid a_n = 0, 1, \ldots, \mathrm{wasc}(a_1 \cdots a_{n-1}) + 1\}$) by applying the pattern-avoiding restrictions of the patterns in P. We arrange the nodes from the left to the right so that if $a_1 \cdots a_{n-1}i$ and $a' = a_1 \cdots a_{n-1}i'$ are children of the same parent $a_1 \cdots a_{n-1}$, then a appears on the left of a' if $i < i'$. Clearly, the cardinality of $A_n(P)$ (resp. $WA_n(P)$) equals the number of nodes in the nth level of $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$).

For a given set of patterns P, let $\mathcal{T}(P; a)$ (resp. $\mathcal{T}'(P; a)$) denote the subtree consisting of the ascent sequence a as the root and its descendants in $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$). For any $a, b \in \mathcal{T}(P)$ (resp. $\mathcal{T}(P)$), we say that the subtrees $\mathcal{T}(P; a)$ and $\mathcal{T}(P; b)$ (resp. $\mathcal{T}'(P; a)$ and $\mathcal{T}'(P; b)$) are isomorphic, and write $\mathcal{T}(P; a) \cong \mathcal{T}(P; b)$ (resp. $\mathcal{T}'(P; a) \cong \mathcal{T}'(P; b)$), if these subtrees are isomorphic in the sense of plane (ordered) trees. We define an equivalence relation on the set of nodes of $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$) as follows. Let a and b two nodes in $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$), we say that a is *equivalent* to b, denoted by $a \sim b$, if and only if $\mathcal{T}(P; a) \cong \mathcal{T}(P; b)$ (resp. $\mathcal{T}'(P; a) \cong \mathcal{T}'(P; b)$). Define $V[P]$ to be the set of all equivalence classes in the quotient set $\mathcal{T}(P)/\sim$ (resp. $\mathcal{T}'(P; a)/\sim$). We will represent each equivalence class $[v]$ by the label of the unique node v which appears on the tree $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$) as the left-most node at the lowest level among all other nodes in the same equivalence class. Let $\mathcal{T}[P]$ (resp. $\mathcal{T}'[P]$) be the same tree $\mathcal{T}(P)$ (resp. $\mathcal{T}'[P]$) where we replace each node a by its equivalence class label.

Define $L = \{000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210\}$ to be the set of all length-3 patterns. Since the number of subsets P of L with $|P| = 4$ is 715, it seems impossible to reach our goals by constructing explicit bijections between classes of ascent sequences or weak ascent sequences. The way out is to combine several steps as follows:

Step 1: We find all the sequences $\{|A_n(P)|\}_{n=1}^{10}$ and $\{|WA_n(P)|\}_{n=1}^{10}$, for all $P \subset L$ with $|P| = 4$. Table 4 in the Appendix below, referring to ascent sequences, divides the 715 4-subsets of L into 74 classes, where the first column of this table assigns the number of the class. Theorem 1 is equivalent to proving that the classes in Table 4 are exactly the A-Wilf-equivalences among quadruples of length-3 patterns. Table 5 in the Appendix below, referring to weak ascent sequences, divides the 715 4-subsets of L into 228 classes, where the first column of this table assigns the number of the class. Theorem 2 is equivalent to proving that the classes in Table 5 are exactly the $W\!A$-Wilf-equivalences among quadruples of length-3 patterns.
Step 2: Let C be any class in Table 4 (Table 5). We say that C is *trivial* if C contains exactly one quadruple. Otherwise, C is *nontrivial*. Since each trivial class in Table 4 (Table 5) forms an A-Wilf-equivalence class ($W\!A$-Wilf-equivalence class), we need to consider only the nontrivial classes in Table 4 (Table 5). There are exactly 34 trivial classes, denoted by T in the first column in Table 4 and there are exactly 134 trivial classes, denoted by T in the first column in Table 5. Thus, it remains to consider $74 - 34 = 40$ nontrivial classes for ascent sequences and $228 - 135 = 93$ nontrivial classes for weak ascent sequences.
Step 3: Let B be any set of patterns in P. We say that B is *reducible* if there exists $C \subsetneqq B$ such that $A_n(B) = A_n(C)$ in case of ascent sequences and $WA_n(B) = WA_n(C)$ in case of weak ascent sequences, for all $n \geq 0$. In this context, we write $C \sim_r B$. Clearly, $C \sim_r B$ implies $C \overset{a}{\sim} B$ and $C \overset{w}{\sim} B$ in the respective cases. The following result is shown in [5].

Theorem 3. *We have*

(1)

- $\{001\} \sim_r \{001, 101\}$,
- $\{001\} \sim_r \{001, 102\}$,
- $\{001\} \sim_r \{001, 201\}$,
- $\{011\} \sim_r \{011, 101\}$,
- $\{011\} \sim_r \{011, 110\}$,
- $\{012\} \sim_r \{012, 102\}$,
- $\{012\} \sim_r \{012, 120\}$,
- $\{021\} \sim_r \{021, 201\}$,
- $\{021\} \sim_r \{021, 210\}$.

(2)

- $\{000,001\} \sim_r \{000,001,100\}$,
- $\{000,011\} \sim_r \{000,011,\tau\}$, *for any* $\tau = 100,201,210$,
- $\{000,012\} \sim_r \{000,012,\tau\}$, *for any* $\tau = 201,210$,
- $\{000,021\} \sim_r \{000,021,100\}$,
- $\{001,010\} \sim_r \{001,010,\tau\}$, *for any* $\tau = 021,100,110,120,210$,
- $\{001,011\} \sim_r \{001,011,021\}$,
- $\{001,012\} \sim_r \{001,012,021\}$,
- $\{001,110\} \sim_r \{001,110,210\}$,
- $\{001,120\} \sim_r \{001,120,210\}$,
- $\{010,011\} \sim_r \{010,011,100\}$,
- $\{010,012\} \sim_r \{010,012,\tau\}$, *for any* $\tau = 101,201$,
- $\{010,021\} \sim_r \{010,021,\tau\}$, *for any* $\tau = 100,101,102,110,120$.

(3)

- $\{001,011,012\} \sim_r \{001,011,012,210\}$,
- $\{001,011,100\} \sim_r \{001,011,100,210\}$,
- $\{001,012,100\} \sim_r \{001,012,100,210\}$.

Step 4: For each nontrivial class either in case of ascent sequences or in case of weak ascent sequences, we apply the following basic algorithm for guessing the generating tree $\mathcal{T}[P]$ or $\mathcal{T}'[P]$, respectively:

(1) Let P be any set of patterns and let D be any positive number (here we use $D = 8$).

(2) We find the first D levels of the generating tree $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$).

(3) By (2), we guess all the succession rules of $\mathcal{T}(P)$ (resp. $\mathcal{T}'(P)$).

(4) Based on (3), we try to prove these succession rules. If we fail, then we increase D by 1 and go back to Step (2). Otherwise, the succession rules of the generating tree $\mathcal{T}[P]$ (resp. $\mathcal{T}'[P]$) are found.

We refer the reader to [7, 8] for many examples of ascent sequences avoiding a triple of length-3 patterns, or weak ascent sequences avoiding either a pair or a triple of length-3 patterns. In the next two sections, we present in tables the succession rules (if found) for each quadruple of patterns in a nontrivial class of ascent sequences, respectively weak ascent sequences.

3. Ascent Sequences

By applying the strategy of the previous section, we obtain Table 1 for ascent sequences. From Table 4, we see that the classes 7, 11, 17, 23-26, 29, 36-41, 43–45, 47-51, 54, 55, 58, 59, 63-68, 70, and 74 are trivial. Using Theorem 3, each reducible quadruple is marked with a star.

Beginning of Table 1

Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
1	{000,001,010,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 011$	
	{000,010,011,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01$	
	{000,001,011,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 010$	$x + 2x^2 + x^3$
2	{000,001,010,011}	$0 \rightsquigarrow 00, 01$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	{001,010,011,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$	$\frac{x}{1-x} + x^2$
3	{000,011,012,100}*		
	{000,011,012,021}*		
	{000,011,012,101}*		
	{000,011,012,102}*		
	{000,011,012,110}*		
	{000,011,012,120}*		
	{000,011,012,201}*		
	{000,011,012,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 001$	
	{000,001,012,110}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 010, 010$	$x + 2x^2 + 2x^3$
4	{000,010,012,021}		
	{000,010,012,100}		
	{000,010,012,101}*		
	{000,010,012,102}*		
	{000,010,012,110}		
	{000,010,012,120}*		
	{000,010,012,201}*		
	{000,010,012,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01$; $01 \rightsquigarrow 011$	
	{000,001,012,100}*		
	{000,001,012,101}*		
	{000,001,012,021}*		
	{000,001,012,102}*		
	{000,001,012,120}*		
	{000,001,012,201}*		
	{000,001,012,210}*	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 010, 011$; $011 \rightsquigarrow 010$	$x + 2x^2 + 2x^3 + x^4$
5	{001,011,012,100}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010$	
	{000,001,011,120}	$a_0 \rightsquigarrow 00, a_1$; $a_1 \rightsquigarrow 010, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	$\frac{x}{1-x} + x^2 + x^3$
6	{000,001,010,021}*		
	{000,001,010,100}*		
	{000,001,010,101}*		
	{000,001,010,102}*		
	{000,001,010,110}*		
	{000,001,010,120}*		
	{000,001,010,201}*		
	{000,001,010,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{000,010,011,021}		
	{000,010,011,100}*		
	{000,010,011,101}*		
	{000,010,011,102}		
	{000,010,011,110}*		
	{000,010,011,120}		
	{000,010,011,201}*		
	{000,010,011,210}*	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow a_1$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	{001,010,011,021}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	$\{001,010,011,100\}^*$ $\{001,010,011,101\}^*$ $\{001,010,011,102\}^*$ $\{001,010,011,110\}^*$ $\{001,010,011,120\}^*$ $\{001,010,011,201\}^*$ $\{001,010,011,210\}^*$	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 00$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	$\{001,010,012,021\}^*$ $\{001,010,012,100\}^*$ $\{001,010,012,101\}^*$ $\{001,010,012,102\}^*$ $\{001,010,012,110\}^*$ $\{001,010,012,120\}^*$ $\{001,010,012,201\}^*$ $\{001,010,012,210\}^*$ $\{001,011,012,021\}^*$ $\{001,011,012,101\}^*$ $\{001,011,012,102\}^*$ $\{001,011,012,110\}^*$ v001,011,012,120$\}^*$ $\{001,011,012,201\}^*$ $\{001,011,012,210\}^*$	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 01$	
	$\{010,011,012,021\}$ $\{010,011,012,100\}^*$ $\{010,011,012,101\}^*$ $\{010,011,012,102\}^*$ $\{010,011,012,110\}^*$ $\{010,011,012,120\}^*$ $\{010,011,012,201\}^*$ $\{010,011,012,210\}$	$0 \rightsquigarrow 0, 01$	
	$\{000,001,011,102\}^*$ $\{000,001,011,100\}^*$ $\{000,001,011,021\}^*$ $\{000,001,011,101\}^*$ $\{000,001,011,110\}^*$ $\{000,001,011,201\}^*$ $\{000,001,011,210\}$	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	$x + \frac{2x^2}{1-x}$
8	$\{000,012,100,101\}$ $\{000,012,021,101\}^*$ $\{000,012,101,102\}^*$ $\{000,012,101,120\}^*$ $\{000,012,101,201\}^*$ $\{000,012,101,210\}^*$	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 010, 001$;$001 \rightsquigarrow 010$	
	$\{000,012,100,110\}$ $\{000,012,021,110\}$ $\{000,012,102,110\}^*$ $\{000,012,110,120\}^*$ $\{000,012,110,201\}^*$ $\{000,012,110,210\}^*$	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 001, 011$;$001 \rightsquigarrow 011$	$x + 2x^2 + 3x^3 + 2x^4$
9	$\{000,011,102,120\}$	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 001$; $a_1 \rightsquigarrow 010, a_2$; $001 \rightsquigarrow a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	$\{001,011,100,120\}$	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 00$; $a_1 \rightsquigarrow 010, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	$\{001,012,100,110\}$	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 011$; $011 \rightsquigarrow 011$	$x + \frac{2x^2}{1-x} + x^3$
10	$\{000,012,021,100\}$ $\{000,012,100,102\}^*$ $\{000,012,100,120\}^*$ $\{000,012,100,201\}^*$		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,012,100,210}* {000,012,021,102}* {000,012,021,120}* {000,012,021,201}* {000,012,021,210}* {000,012,102,120}* {000,012,102,201}* {000,012,102,210}* {000,012,120,201}* {000,012,120,210}* {000,012,201,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001$; $01 \rightsquigarrow 001, 001$; $001 \rightsquigarrow 0011$	$x + 2x^2 + 3x^3 + 3x^4$
12	{000,011,021,102} {000,011,100,102}* {000,011,101,102}* {000,011,102,110}* {000,011,102,201}* {000,011,102,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = 0a_m$	
	{000,011,100,120}* {000,011,021,120} {000,011,101,120}* {000,011,110,120}* {000,011,120,201}* {000,011,120,210}*	$a_0 \rightsquigarrow 00, a_1$; $a_1 \rightsquigarrow 001, a_2$; $00 \rightsquigarrow 001$; $001 \rightsquigarrow a_2$, where $a_m = 01 \cdots m$	
	{001,011,100,102}* {001,011,021,100}* {001,011,021,120}* {001,011,100,101}* {001,011,100,110}* {001,011,100,201}* {001,011,100,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_0 \rightsquigarrow b_0$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{001,011,102,120}* {001,011,101,120}* {001,011,110,120}* {001,011,120,201}* {001,011,120,210}*	$a_0 \rightsquigarrow 00, a_1$; $a_1 \rightsquigarrow 010, a_2$; $00 \rightsquigarrow 00$; $010 \rightsquigarrow 010$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	{001,012,100,101}* {001,012,021,100}* {001,012,100,102}* {001,012,100,120}* {001,012,100,201}* {001,012,100,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$	
	{001,012,101,110}* {001,012,021,110} {001,012,102,110}* {001,012,110,120}* {001,012,110,201}* {001,012,110,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 010$	
	{011,012,021,100} {011,012,100,101}* {011,012,100,102}* {011,012,100,110}* {011,012,100,120}* {011,012,100,201} {011,012,100,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010$	
	{000,001,110,120}		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,001,021,110}	$a_0 \rightsquigarrow 00, a_1$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$	$x + 2x^2 + \frac{3x^3}{1-x}$
13	{000,001,101,120}* {000,001,100,120}* {000,001,102,120}* {000,001,120,201}* {000,001,120,210}*	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	
	{000,001,021,102}* {000,001,021,101}* {000,001,021,100}* {000,001,021,210}* {000,001,021,201}*	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	$x + 2x^2 + 3x^3 + \frac{4x^4}{1-x}$
14	{000,011,021,100}* {000,011,100,101}* {000,011,100,110}* {000,011,100,201}* {000,011,100,210}* {000,011,021,101}* {000,011,021,110}* {000,011,021,201}* {000,011,021,210}* {000,011,101,110}* {000,011,101,201}* {000,011,101,210}* {000,011,110,201}* {000,011,110,210}* {000,011,201,210}* {001,011,021,101}* {001,011,021,110}* {001,011,021,201}* {001,011,021,210}* {001,011,101,110}* {001,011,101,201}* {001,011,101,210}* {001,011,110,201}* {001,011,110,210}* {001,011,201,210}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = 0a_m$	
	{001,010,021,100}* {001,010,021,101}* {001,010,021,102}* {001,010,021,110}* {001,010,021,120}* {001,010,021,201}* {001,010,021,210}* {001,010,100,101}* {001,010,100,102}* {001,010,100,110}* {001,010,100,120}* {001,010,100,201}* {001,010,100,210}* {001,010,101,102}* {001,010,101,110}* {001,010,101,120}* {001,010,101,201}* {001,010,101,210}* {001,010,102,110}* {001,010,102,120}* {001,010,102,201}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{001,010,102,210}* {001,010,110,120}* {001,010,110,201}* {001,010,110,210}* {001,010,120,201}* {001,010,120,210}* {001,010,201,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,011,021,102}* {001,011,101,102}* {001,011,102,110}* {001,011,102,201}* {001,011,102,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{001,012,021,101}* {001,012,101,102}* {001,012,101,120}* {001,012,101,201}* {001,012,101,210}* {001,012,021,102}* {001,012,021,120}* {001,012,021,201}* {001,012,021,210}* {001,012,102,120}* {001,012,102,201}* {001,012,102,210}* {001,012,120,201}* {001,012,120,210}* {001,012,201,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$	
	{010,011,021,100}* {010,011,021,101}* {010,011,021,102}* {010,011,021,110}* {010,011,021,120}* {010,011,021,201}* {010,011,021,210}* {010,011,100,101}* {010,011,100,102}* {010,011,100,110}* {010,011,100,120}* {010,011,100,201}* {010,011,100,210}* {010,011,101,102}* {010,011,101,110}* {010,011,101,120}* {010,011,101,201}* {010,011,101,210}* {010,011,102,110}* {010,011,102,120} {010,011,102,201} {010,011,102,210} {010,011,110,120}* {010,011,110,201}* {010,011,110,210}* {010,011,120,201} {010,011,120,210} {010,011,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$	
	{010,012,021,100}* {010,012,021,101}* {010,012,021,102}* {010,012,021,110}* {010,012,021,120}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{010,012,021,201}*		
	{010,012,021,210}*		
	{010,012,100,101}*		
	{010,012,100,102}*		
	{010,012,100,110}		
	{010,012,100,120}*		
	{010,012,100,201}*		
	{010,012,100,210}		
	{010,012,101,102}*		
	{010,012,101,110}*		
	{010,012,101,120}*		
	{010,012,101,201}*		
	{010,012,101,210}*		
	{010,012,102,110}*		
	{010,012,102,120}*		
	{010,012,102,201}*		
	{010,012,102,210}*		
	{010,012,110,120}*		
	{010,012,110,201}*		
	{010,012,110,210}		
	{010,012,120,201}*		
	{010,012,120,210}*		
	{010,012,201,210}*		
	{011,012,021,101}*		
	{011,012,021,102}*		
	{011,012,021,110}*		
	{011,012,021,120}*		
	{011,012,021,201}*		
	{011,012,021,210}*		
	{011,012,101,102}*		
	{011,012,101,110}*		
	{011,012,101,120}*		
	{011,012,101,201}*		
	{011,012,101,210}*		
	{011,012,102,110}*		
	{011,012,102,120}*		
	{011,012,102,201}*		
	{011,012,102,210}*		
	{011,012,110,120}*		
	{011,012,110,201}*		
	{011,012,110,210}*		
	{011,012,120,201}*		
	{011,012,120,210}*		
	{011,012,201,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 01$	
	{000,001,102,110}*		
	{000,001,101,110}*		
	{000,001,100,110}*		
	{000,001,110,201}*		
	{000,001,110,210}*	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	$\frac{x}{(1-x)^2}$
15	{000,001,102,210}*		
	{000,001,101,210}*		
	{000,001,100,210}*		
	{000,001,201,210}*	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow (b_m)^m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$, $c_m = a_m m$	$\frac{x(1+x^3)}{(1-x)^2}$
16	{000,010,021,100}*		
	{000,010,021,101}*		
	{000,010,021,102}*		
	{000,010,021,110}*		
	{000,010,021,120}*		
	{000,010,021,201}*		
	{000,010,021,210}*		
	{000,010,100,101}		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,010,100,102} {000,010,100,110} {000,010,100,120} {000,010,100,201} {000,010,100,210} {000,010,101,102} {000,010,101,110} {000,010,101,120} {000,010,101,201} {000,010,101,210} {000,010,102,110} {000,010,102,120} {000,010,102,201} {000,010,102,210} {000,010,110,120} {000,010,110,201} {000,010,110,210} {000,010,120,201} {000,010,120,210} {000,010,201,210}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{000,001,100,102}* {000,001,101,102}* {000,001,100,101}* {000,001,102,201}* {000,001,101,201}* {000,001,100,201}*	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j-1}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$	$\frac{x(1+x)}{1-x-x^2}$
18	{011,100,102,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$	
	{012,100,101,110}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 011$; $011 \rightsquigarrow 011$	
	{001,100,110,120} {001,021,100,110}	$a_0 \rightsquigarrow c_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	
	{001,021,100,120} {001,021,110,120}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	$\frac{x(1+x^2-x^3)}{(1-x)^2}$
19	{011,021,100,102} {011,100,101,102}* {011,100,102,110}* {011,100,102,201} {011,100,102,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m 0$	
	{011,021,100,120} {011,021,102,120} {011,100,101,120}* {011,100,110,120}* {011,100,120,201} {011,100,120,210} {011,101,102,120}* {011,102,110,120}* {011,102,120,201} {011,102,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_2$; $010 \rightsquigarrow a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$	
	{012,021,100,101} {012,100,101,102}* {012,100,101,120}* {012,100,101,201} {012,100,101,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 01$	
	{012,021,100,110} {012,100,102,110}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{012,100,110,120}* {012,100,110,201} {012,100,110,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$	
	{012,021,101,110} {012,101,102,110}* {012,101,110,120}* {012,101,110,201} {012,101,110,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 010$; $010 \rightsquigarrow 010$	
	{001,100,102,120}* {001,100,101,120}* {001,100,120,201}* {001,100,120,210}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	
	{001,102,110,120}* {001,101,110,120}* {001,110,120,201}* {001,110,120,210}*	$a_0 \rightsquigarrow 00, a_1$; $00 \rightsquigarrow 00$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$	
	{001,100,102,110}* {001,100,101,110}* {001,100,110,201}* {001,100,110,210}*	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{001,021,101,120}* {001,021,102,120}* {001,021,120,201}* {001,021,120,210}*	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow 010, b_1, a_2$; $010 \rightsquigarrow 010$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,021,100,102}* {001,021,100,101}* {001,021,100,210}* {001,021,100,201}*	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m 0$	
	{001,021,102,110}* {001,021,101,110}* {001,021,110,210}* {001,021,110,201}*	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$	$\frac{x(1+x^2)}{(1-x)^2}$
20	{001,021,201,210}* {001,101,102,120}* {001,021,101,102}* {001,021,102,210}* {001,021,102,201}* {001,021,101,201}* {001,021,101,210}*	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_m \rightsquigarrow b_m, c_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$, $c_m = a_m m$	
	{011,021,100,101}* {011,021,100,110}* {011,021,100,201}* {011,021,100,210}* {011,100,101,110}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{011,100,101,201}* {011,100,101,210}* {011,100,110,201}* {011,100,110,210}* {011,100,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_2$; $010 \rightsquigarrow b_2$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = 01023 \cdots m$	
	{011,021,101,102}* {011,021,102,110}* {011,021,102,201}* {011,021,102,210}* {011,101,102,110}* {011,101,102,201}* {011,101,102,210}* {011,102,110,201}* {011,102,110,210}* {011,102,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01 \cdots m$, $b_m = a_m 0$	
	{012,021,100,102}* {012,021,100,120}* {012,021,100,201}* {012,021,100,210}* {012,100,102,120}* {012,100,102,201}* {012,100,102,210}* {012,100,120,201}* {012,100,120,210}* {012,100,201,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$	
	{012,021,101,102}* {012,021,101,120}* {012,021,101,201}* {012,021,101,210}* {012,101,102,120}* {012,101,102,201}* {012,101,102,210}* {012,101,120,201}* {012,101,120,210}* {012,101,201,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 010, 01$; $010 \rightsquigarrow 010$	
	{011,021,101,120}* {011,021,110,120}* {011,021,120,201}* {011,021,120,210}* {011,101,110,120}* {011,101,120,201}* {011,101,120,210}* {011,110,120,201}* {011,110,120,210}* {011,120,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow a_1, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01 \cdots m$	
	{012,021,102,110}* {012,021,110,120}* {012,021,110,201}* {012,021,110,210}* {012,102,110,120}* {012,102,110,201}* {012,102,110,210}* {012,110,120,201}* {012,110,120,210}* {012,110,201,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 01, 011$; $011 \rightsquigarrow 011$	
	{001,101,102,110}* {001,102,110,201}*		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{001,102,110,210}* {001,101,110,201}* {001,101,110,210}* {001,110,201,210}*	$a_m \rightsquigarrow (b_m)^{m+1}, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m0$	
	{001,101,120,201}* {001,101,120,210}* {001,102,120,201}* {001,102,120,210}* {001,120,201,210}*	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow b_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, b_m$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	
	{001,100,102,210}* {001,100,101,210}* {001,100,201,210}*	$a_m \rightsquigarrow (b_m)^m, c_m, a_{m+1}$; $c_m \rightsquigarrow (b_m)^m, c_m$, where $a_m = 01\cdots m$, $b_m = a_m0$, $c_m = a_m m$	$\frac{x(1-x+x^2)}{(1-x)^3}$
21	{000,101,102,120} {000,102,110,120} {000,021,101,102}	$a_0 \rightsquigarrow c_0, a_1$; $c_0 \rightsquigarrow d_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow b_m, d_{m+1}$; $d_m \rightsquigarrow e_m, a_{m+1}$; $e_m \rightsquigarrow d_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$, $d_m = c_m(m+1)$, $e_m = c_{m-1}mm$	
	{000,021,101,120}	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow 010, 001, a_2$; $b_0 \rightsquigarrow 001$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_m \rightsquigarrow a_{m+1}$; $001 \rightsquigarrow 019, a_2$; $010 \rightsquigarrow a_2$, where $a_m = 01\cdots m$, $b_m = a_m m$	$\frac{x(1+x+x^2+x^3)}{1-x-x^2}$
22	{001,100,101,102}* {001,100,102,201}* {001,100,101,201}*	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,m} \rightsquigarrow b_{m,0}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j-1}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$	
	{000,101,102,110}	$a_m \rightsquigarrow (b_m)^m, c_{m,m}, a_{m+1}$; $c_{m,j} \rightsquigarrow (b_m)^{m-1-j}, c_{m,m}, c_{m+1,j}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $c_{m,j} = a_j j(j+1)\cdots m$	
	{000,021,101,110}	$a_0 \rightsquigarrow b_0, a_1$; $b_0 \rightsquigarrow c_1$; $a_m \rightsquigarrow (b_m)^2, a_{m+1}$; $b_m \rightsquigarrow c_{m+1}$; $c_m \rightsquigarrow b_m, c_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $c_m = 0a_m$	$\frac{x}{(1-x)(1-x-x^2)}$
27	{000,021,102,201}* {000,021,102,210}* {000,021,100,102}*	$a_0 \rightsquigarrow c_0, a_1$; $c_0 \rightsquigarrow 001$; $c_1 \rightsquigarrow 0110, a_2$; $c_2 \rightsquigarrow 0101, a_3$; $010 \rightsquigarrow 0101$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow b_m, a_{m+1}$; $001 \rightsquigarrow d_1, e_2$; $d_m \rightsquigarrow e_{m+1}$; $e_m \rightsquigarrow d_m, e_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $c_m = a_m m$, $d_m = 0a_m$, $e_m = d_m m$	$\frac{x(1+x+x^2+2x^3-x^4-x^5)}{1-x-x^2}$
28	{000,100,102,120} {000,102,120,201}		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,102,120,210}	$a_0 \rightsquigarrow c_0, a_1$; $c_0 \rightsquigarrow e_0$; $c_1 \rightsquigarrow 0110, e_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $c_m \rightsquigarrow d_{m-1}, e_m$; $b_m \rightsquigarrow d_m$; $e_m \rightsquigarrow f_m, a_{m+2}$; $f_m \rightsquigarrow e_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$, $d_m = b_m m$, $e_m = c_m(m+1)$, $f_m = e_m(m+1)$	$\frac{x(1+x+x^2+2x^3)}{1-x-x^2}$
30	{000,021,120,201}* {000,021,120,210}* {000,021,100,120}*	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow 010, 001, a_2$; $a_m \rightsquigarrow b_m, a_{m+1}$; $b_0 \rightsquigarrow 001$; $010 \rightsquigarrow c_1, a_2$; $001 \rightsquigarrow 0011, a_2$; $0011 \rightsquigarrow a_2$; $b_m \rightsquigarrow d_{m+1}$; $c_1 \rightsquigarrow c_2, a_3$; $c_m \rightsquigarrow d_m, c_{m+1}, a_{m+2}$; $d_m \rightsquigarrow c_{m+1}, a_{m+2}$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = 01a_m$, $d_m = c_m m$	$\frac{x(1-x^2-2x^4-x^5)}{(1-x-x^2)^2}$
31	{000,100,102,110} {000,102,110,210}	$a_m \rightsquigarrow (c_m)^m, b_{m,m}, a_{m+1}$; $b_{m,m} \rightsquigarrow b_{m+1,m}$; $b_{m,j} \rightsquigarrow (c_m)^{m-1-j}, b_{m,m}, b_{m+1,j}$; $c_m \rightsquigarrow d_m$, where $a_m = 01 \cdots m$, $b_{m,j} = a_j j(j+1) \cdots m$, $c_m = a_m 0$, $d_m = a_m 01$	$\frac{x(1+x^3)}{(1-x)(1-x-x^2)}$
32	{001,101,102,210}* {001,102,201,210}* {001,101,201,210}*	$a_m \rightsquigarrow (c_m)^m, b_m, a_{m+1}$; $b_m \rightsquigarrow (c_m)^m, b_m$; $c_m \rightsquigarrow c_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m 0$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
33	{000,021,101,201}* {000,021,101,210}* {000,021,100,101}*	$a_0 \rightsquigarrow b_0, a_0$; $b_0 \rightsquigarrow d_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m, a_{m+1}$; $c_m \rightsquigarrow d_{m+1}$; $d_m \rightsquigarrow c_m, d_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m 0$, $d_m = 0a_m$	
	{000,021,110,210}* {000,021,110,201}* {000,021,100,110}	$a_0 \rightsquigarrow b_0, a_0$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow f_{m+1}$; $c_m \rightsquigarrow d_m, f_{m+1}$; $d_m \rightsquigarrow e_m, f_{m+2}$; $e_m \rightsquigarrow d_m, e_{m+1}, f_{m+2}$; $f_m \rightsquigarrow b_m, f_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_m m$, $c_m = a_m 0$, $d_m = a_m 0m$, $e_m = 01a_m$, $f_m = 0a_m$	
	{000,101,102,210}	$a_m \rightsquigarrow (d_m)^m, c_{m,-1}, a_{m+1}$; $b_{m,j} \rightsquigarrow (d_m)^{m-1-j}, c_{m,j}, b_{m+1,j}$; $c_{m,j} \rightsquigarrow (d_m)^{m-1-j}, b_{m+1,j+1}$, where $a_m = 01 \cdots m$, $b_{m,j} = a_j j(j+1) \cdots m$, $c_{m,j} = b_{m,j} m$	$\frac{x(1-x^2)}{(1-x-x^2)^2}$
34	{000,100,101,120} {000,101,120,201}		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{000,101,120,210}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_0 \rightsquigarrow 001$; $001 \rightsquigarrow c_1, a_2$; $b_m \rightsquigarrow c_m, d_{m+1}$; $c_m \rightsquigarrow d_{m+1}$; $d_m \rightsquigarrow c_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$, $d_m = c_m(m+1)$	$\frac{x(1+x+x^2+x^3)}{1-x-x^2-x^3-x^4}$
35	{001,101,102,201}*	$a_m \rightsquigarrow b_{m,1}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$	
	{010,021,100,101}*		
	{010,021,100,102}*		
	{010,021,100,110}*		
	{010,021,100,120}*		
	{010,021,100,201}*		
	{010,021,100,210}*		
	{010,021,101,102}*		
	{010,021,101,110}*		
	{010,021,101,120}*		
	{010,021,101,201}*		
	{010,021,101,210}*		
	{010,021,102,110}*		
	{010,021,102,120}*		
	{010,021,102,201}*		
	{010,021,102,210}*		
	{010,021,110,120}*		
	{010,021,110,201}*		
	{010,021,110,210}*		
	{010,021,120,201}*		
	{010,021,120,210}*		
	{010,021,201,210}*		
	{010,100,101,102}		
	{010,100,101,110}		
	{010,100,101,120}		
	{010,100,101,201}		
	{010,100,101,210}		
	{010,100,102,110}		
	{010,100,102,120}		
	{010,100,102,201}		
	{010,100,102,210}		
	{010,100,110,120}		
	{010,100,110,201}		
	{010,100,110,210}		
	{010,100,120,201}		
	{010,100,120,210}		
	{010,100,201,210}		
	{010,101,102,110}		
	{010,101,102,120}		
	{010,101,102,201}		
	{010,101,102,210}		
	{010,101,110,120}		
	{010,101,110,201}		
	{010,101,110,210}		
	{010,101,120,201}		
	{010,101,120,210}		
	{010,101,201,210}		
	{010,102,110,120}		
	{010,102,110,201}		
	{010,102,110,210}		

Continuation of Table 1

Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{010,102,120,201} {010,102,120,210} {010,102,201,210} {010,110,120,201} {010,110,120,210} {010,110,201,210} {010,120,201,210} {011,021,101,110} {011,021,101,201}* {011,021,101,210}* {011,021,110,201}* {011,021,110,210}* {011,021,201,210}* {011,101,110,201}* {011,101,110,210}* {011,101,201,210}* {011,110,201,210}*	$a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{012,021,102,120}* {012,021,102,201}* {012,021,102,210}* {012,021,120,201}* {012,021,120,210}* {012,021,201,210}* {012,102,120,201}* {012,102,120,210}* {012,102,201,210}* {012,120,201,210}*	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 01, 01$	
	{000,100,101,102} {000,101,102,201}	$a_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m}, a_{m+1}$; $b_{m,m} \rightsquigarrow b_{m,0}, \ldots, b_{m,m-1}, c_{m+1,0}$; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j-1}$; $c_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,m-2-j}$, $d_{m,j}, c_{m+1,j}$; $d_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,m-2-j}$, $d_{m+1,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_m j$, $c_{m,j} = 0011\cdots jj(j+1)\cdots m$, $d_{m,j} = c_{m,j}m$	
	{000,100,101,110} {000,101,110,201} {000,101,110,210}	$a_m \rightsquigarrow (b_{m,0})^{m+1}, a_{m+1}$; $b_{m,j} \rightsquigarrow (b_{m,0})^{m-j}, b_{m+1,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_j 0(j+1)\cdots m$	$\frac{x}{1-2x}$
42	{000,021,100,201}* {000,021,100,210}* {000,021,201,210}*	$\overset{a}{\sim} \{000, 021\}$	
46	{000,101,201,210} {000,100,101,210}	$a_0 \rightsquigarrow b_0, a_1$; $a_2 \rightsquigarrow c_{1,1}, c_{1,0}, a_2$; $a_m \rightsquigarrow (c_{m,m})^m, b_m, a_{m+1}$; $b_m \rightsquigarrow (c_{m,m})^m, c_{m+1,0}$; $c_{m,m} \rightsquigarrow c_{m+1,m}$; $c_{m,m-1} \rightsquigarrow c_{m,m}, c_{m+1,m-1}$; $c_{m,j} \rightsquigarrow (c_{m,m})^{m-1-j}, d_{m,j}, c_{m+1,j}$; $d_{m,j} \rightsquigarrow (c_{m,m})^{m-2-j}, c_{m+1,j+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_{m,j} = 01\cdots j0(j+1)\cdots m$, $d_{m,j} = c_{m,j}m$	$\frac{x(1-x^2)}{1-2x-x^2+x^3}$
52	{021,100,102,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_1, a_2$; $010 \rightsquigarrow 0101$; $0101 \rightsquigarrow 0101$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{021,101,102,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_1, a_2$; $010 \rightsquigarrow 010$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{021,102,110,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, 011, a_2$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$; $011 \rightsquigarrow 011, a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{021,101,102,110}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_m, b_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = 0112\cdots m$, $c_m = a_m0$	$\frac{x(1-3x+4x^2-3x^3)}{(1-x)^3(1-2x)}$
53	{100,101,102,120} {021,100,101,102}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$	
	{100,102,110,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_{2m+1} \rightsquigarrow c_{2m+1}, b_{2m+1}, a_{2m+2}$; $a_{2m+2} \rightsquigarrow d_{2m+1}, b_{2m+2}, a_{2m+3}$; $b_m \rightsquigarrow b_m, a_{m+1}$; $c_{2m+1} \rightsquigarrow c_{2m+1}$; $d_{2m+1} \rightsquigarrow c_{2m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$; $c_m = a_m(m-1)$, $d_m = c_m m$	
	{101,102,110,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_m, a_{m+1}$; $c_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$	
	{100,101,102,110}	$a_m \rightsquigarrow (c_m)^m, b_{m,m}, a_{m+1}$; $b_{m,j} \rightsquigarrow (c_m)^{m-j}, b_{m,m}, b_{m+1,m}$, where $a_m = 01\cdots m$, $b_{m,j} = a_j j(j+1)\cdots m$, $c_m = a_m0$	
	{021,100,101,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_1, a_2$; $010 \rightsquigarrow a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{021,101,110,120}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow (010)^2, a_2$; $010 \rightsquigarrow 010, a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{021,100,101,110}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow b_1, 011, a_2$; $011 \rightsquigarrow 011, c_2$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow c_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $c_m = 0102\cdots m$	$\frac{x(1-x+x^2)}{(1-x)(1-2x)}$
56	{021,102,110,201}* {021,102,110,210}*	$\stackrel{a}{\sim}$021,102,110	
57	{021,100,102,201}* {021,100,102,210}*	$\stackrel{a}{\sim}$021,100,102	
60	{100,102,120,201} {100,102,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_{2m+1} \rightsquigarrow c_{2m+1}, a_{2m+1}, a_{2m+2}$; $a_{2m+2} \rightsquigarrow d_{2m+1}, a_{2m+2}, a_{2m+3}$; $c_{2m+1} \rightsquigarrow d_{2m+1}$; $d_{2m+1} \rightsquigarrow d_{2m+1}$, where $a_m = 01\cdots m$, $c_m = a_m(m-1)$, $d_m = c_m m$	
	{101,102,120,201} {101,102,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$	
	{101,102,110,201} {101,102,110,210}	$a_m \rightsquigarrow (b_m)^m, c_{m,m}, a_{m+1}$; $b_m \rightsquigarrow b_m$; $c_{m,j} \rightsquigarrow (b_m)^{m-j}, c_{m,m}, c_{m+1,j}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $b_{m,j} = a_j j(j+1)\cdots m$	
	{102,110,120,201}		

Continuation of Table 1			
Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{102,110,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_m, a_{m+1}$; $c_m \rightsquigarrow c_m, d_m$; $d_m \rightsquigarrow d_m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = a_m(m-1)$, $d_m = c_m m$	
	{021,102,120,201}* {021,102,120,210}* {021,101,120,201}* {021,101,120,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_1, a_2$; $010 \rightsquigarrow 010, a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$, where $a_m = 01\cdots m$	
	{021,101,102,201}* {021,101,102,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m 0$	$\frac{x(1-2x+2x^2)}{(1-x)^2(1-2x)}$
61	{100,101,102,210}	$a_m \rightsquigarrow (b_m)^m, a_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{021,100,120,201}* {021,100,120,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, a_1, a_2$; $010 \rightsquigarrow c_1, a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$; $c_m \rightsquigarrow c_m, c_{m+1}, a_{m+2}$, where $a_m = 01\cdots m$, $c_m = 01a_m$	
	{021,101,110,201}* {021,101,110,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_m, b_{m+1}$; $c_m \rightsquigarrow c_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = 0112\cdots m$, $c_m = a_m 0$	
	{021,100,101,201}* {021,100,101,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow c_m, a_m, a_{m+1}$; $c_m \rightsquigarrow d_{m+1}$; $d_m \rightsquigarrow d_m, d_{m+1}$, where $a_m = 01\cdots m$, $c_m = a_m 0$, $d_m = 0102\cdots m$	
	{021,100,110,201}* {021,100,110,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow c_1, 011, a_2$; $011 \rightsquigarrow 011, b_2$; $a_m \rightsquigarrow c_m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_m, b_{m+1}$; $c_m \rightsquigarrow b_m, b_{m+1}$, where $a_m = 01\cdots m$, $c_m = a_m 0$, $b_m = 0102\cdots(m-2)m$	
	{021,110,120,201}* {021,110,120,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow 010, 011, a_2$; $010 \rightsquigarrow 010, c_1, a_2$; $011 \rightsquigarrow 011, a_2$; $a_m \rightsquigarrow a_m, a_{m+1}$; $c_m \rightsquigarrow c_m, c_{m+1}, a_{m+2}$, where $a_m = 01\cdots m$, $c_m = 01a_m$	$\frac{x(1-x)^2}{(1-2x)^2}$
62	{100,101,120,201} {100,101,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$; $b_m \rightsquigarrow c_{m+1}$; $c_m \rightsquigarrow c_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = a_{m-1}(m-2)m$	
	{101,110,120,201} {101,110,120,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m, c_{m+1}$; $c_m \rightsquigarrow c_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = a_{m-1}(m-2)m$	
	{100,101,110,201}		

Continuation of Table 1

Class	B quadruple	Rules of $\mathcal{T}(B)$	$G_B(x)$/Reference
	{100,101,110,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow b_{1,1}, c_1, a_2$, $c_1 \rightsquigarrow c_1, c_2$, $a_m \rightsquigarrow (b_{m,m})^m, b_{m,m-1}, a_{m+1}$; $b_{m,m} \rightsquigarrow b_{m+1,m}$; $b_{m,j} \rightsquigarrow (b_{m,m})^{m-1-j}, b_{m,m-1}, b_{m+1,j}$; $c_m \rightsquigarrow (b_{m,m})^{m-1}, b_{m,m-1}, c_{m+1}$, where $a_m = 01\cdots m$, $b_{m,j} = a_j0(j+1)\cdots m$, $c_m = 0112\cdots m$	$\frac{x(1-x+x^2)}{1-3x+2x^2-x^3}$
69	{101,102,201,210}	$a_m \rightsquigarrow (b_m)^m, a_m, a_{m+1}$; $b_m \rightsquigarrow b_m$, where $a_m = 01\cdots m$, $b_m = a_m0$	$\frac{x(1-3x+3x^2)}{(1-x)(1-2x)^2}$
	{102,120,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_{2m} \rightsquigarrow c_{2m-1}, a_m, a_{m+1}$; $a_{2m+1} \rightsquigarrow b_{2m+1}, a_m, a_{m+1}$; $b_m \rightsquigarrow b_m, c_m$; $c_m \rightsquigarrow (c_m)^2$, where $a_m = 01\cdots m$, $b_m = a_m(m-1)$, $c_m = b_m m$	
71	{021,100,201,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$; $b_m \rightsquigarrow d_m, c_{m+1}$; $c_m \rightsquigarrow c_m, c_{m+1}$; $d_m \rightsquigarrow d_m, d_{m+1}, c_{m+2}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $d_m = 01a_m$, $c_m = d_{m-2}m$	$\frac{x(1-4x+5x^2-x^3)}{(1-2x)^3}$
	{021,110,201,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_1 \rightsquigarrow b_1, 011, a_2$; $011 \rightsquigarrow 011, 0112$; $0112 \rightsquigarrow 0112, c_2$; $a_m \rightsquigarrow b_m, c_m, a_{m+1}$; $b_m \rightsquigarrow b_m, d_m, a_{m+1}$; $c_m \rightsquigarrow c_m, c_{m+1}$; $d_m \rightsquigarrow d_m, d_{m+1}, c_{m+2}$, where $a_m = 01\cdots m$, $b_m = a_m0$, $d_m = 01a_m$, $c_m = d_{m-2}m$	
72	{100,101,201,210}	$a_m \rightsquigarrow (b_{m,m})^m, a_m, a_{m+1}$; $b_{m,m} \rightsquigarrow b_{m+1,m}$; $b_{m,j} \rightsquigarrow (b_{m,m})^{m-1-j}, b_{m,j}, b_{m+1,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_j0(j+1)\cdots m$	$\frac{x(1-x)}{1-3x+x^2}$
	{101,110,201,210}	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow (b_{m,m})^{m+1}, a_{m+1}$; $b_{m,j} \rightsquigarrow (b_{m,m})^{m+1-j}, b_{m+1,j}$, where $a_m = 01\cdots m$, $b_{m,j} = a_j0(j+1)\cdots m$	
	{021,101,201,210}*	$a_0 \rightsquigarrow a_0, a_1$; $a_m \rightsquigarrow b_m, a_m, a_{m+1}$; $b_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m0$	
	{021,120,201,210}*	$0 \rightsquigarrow 0, b_2$; $b_{2m} \rightsquigarrow b_{2m+1}, b_{2m}, a_{2,m}, a_{3,m}, \ldots, a_{m+1,1}$; $b_{2m+1} \rightsquigarrow b_{2m+1}, b_{2m+2}, a_{2,m}, a_{3,m}, \ldots, a_{m+1,1}$, where $a_{m,j} = (01)^j2\cdots m$, $b_{2m} = (01)^m$, $b_{2m+1} = b_{2m}0$	
73	{100,120,201,210}	?	See note below
	{110,120,201,210}	?	

End of Table 1

Table 1: Rules of generating trees for ascent sequences avoiding a quadruple of patterns of length three.

Class 73: Note that only for Class 73 did we fail to find the succession rules of the generating trees $\mathcal{T}(100,120,201,210)$ and $\mathcal{T}(110,120,201,210)$. But the equivalence $\{100,120,201,210\} \overset{a}{\sim} \{110,120,201,210\}$ follows from the bijection in Section 4.8

of [7] where it is shown that $\{110,120,201\} \overset{a}{\sim} \{100,120,201\}$. Clearly, this bijection preserves 210 avoiding sequences. □

4. Weak Ascent Sequences

For weak ascent sequences, we similarly obtain Table 2. From Table 5, the classes 3, 7, 9, 13, 20-26, 28-39, 41, 50, 53, 54,59, 60, 62, 66, 71, 72, 74, 75, 77, 78, 80, 81, 84, 85, 91, 96, 98, 100, 103, 104, 106-108, 110-112, 116-118, 122, 124, 126, 128-144, 146, 147, 148, 150-161, 163-172, 175-177, 179-184, 187-189, 198-200, 202-204, 207-210, 212, 213, 216-218, 220-225, and 228 are trivial. Again, using Theorem 3, each reducible quadruple is marked with a star. Note that Class 215 is the only unresolved case, leading to the conclusion of Theorem 2 that $228 \leq \mathrm{WAS}_4 \leq 229$.

Beginning of Table 2

Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
1	{000,001,010,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 011$	$x + 2x^2 + x^3$
	{000,001,011,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 00$	
2	{000,001,010,011}	$a_0 \rightsquigarrow 00, a_1$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$	$\frac{x}{1-x} + x^2$
	{001,010,011,012}	$a_1 \rightsquigarrow a_2, 01$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 0^m$	
4	{000,010,011,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $002 \rightsquigarrow 001$	$x + 2x^2 + 2x^3 + x^4$
	{000,001,012,100} {000,001,012,101}* {000,001,012,021}* {000,001,012,102}* {000,001,012,120}* {000,001,012,201}* {000,001,012,210}*	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 00, 011$; $011 \rightsquigarrow 0110$	
5	{001,011,012,100}	$a_1 \rightsquigarrow a_2, 01$; $a_m \rightsquigarrow a_{m+1}$; $01 \rightsquigarrow 010$, where $a_m = 0^m$	$x^2 + x^3 + \frac{x}{1-x}$
	{000,001,011,120}	$a_0 \rightsquigarrow 00, a_1$; $a_1 \rightsquigarrow 00, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 01\cdots m$	
6	{000,001,010,021}* {000,001,010,100}* {000,001,010,101}* {000,001,010,102}* {000,001,010,110}* {000,001,010,120}* {000,001,010,201}* {000,001,010,210}*	$a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,010,011,021}* {001,010,011,100}* {001,010,011,101}* {001,010,011,102}* {001,010,011,110}* {001,010,011,120}* {001,010,011,201}* {001,010,011,210}* {001,010,012,021}* {001,010,012,100}* {001,010,012,101}* {001,010,012,102}*		

Continuation of Table 2

Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{001,010,012,110}* {001,010,012,120}* {001,010,012,201}* {001,010,012,210}* {001,011,012,021}* {001,011,012,101}* {001,011,012,102}* {001,011,012,110}* {001,011,012,120}* {001,011,012,201} {001,011,012,210}*	$a_1 \rightsquigarrow a_2, a_2$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 0^m$	$x + \frac{2x^2}{1-x}$
	{000,001,011,102}* {000,001,011,100}* {000,001,011,021}* {000,001,011,101}* {000,001,011,110}* {000,001,011,201}* {000,001,011,210}*	$a_0 \rightsquigarrow 00, a_1$; $a_1 \rightsquigarrow 00, a_2$; $a_m \rightsquigarrow b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m0$	
8	{000,011,012,100}* {000,011,012,101}* {000,011,012,102}* {000,011,012,110}* {000,011,012,120}* {000,011,012,201}* {000,011,012,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010$; $002 \rightsquigarrow 001$	$x + 2x^2 + 3x^3 + x^4$
10	{001,011,100,120} {001,012,100,110}	$a_1 \rightsquigarrow a_2, 01$; $a_m \rightsquigarrow a_{m+1}$; $01 \rightsquigarrow 010, a_3$, where $a_m = 0^m$	$x + x^3 + \frac{2x^2}{1-x}$
11	{000,010,012,100}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 011$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 011, 0022$	$x + 2x^2 + 3x^3 + 3x^4 + x^5$
	{000,010,012,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 011$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 001, 0011$	
12	{000,010,012,101}* {000,010,012,102}* {000,010,012,120}* {000,010,012,201}* {000,010,012,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 011$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 001, 0022$; $0022 \rightsquigarrow 00221$; $00221 \rightsquigarrow 002211$	$x + 2x^2 + 3x^3 + 3x^4 + 2x^5 + x^6$
14	{001,011,100,102} {001,011,021,100}* {001,011,100,101}* {001,011,100,110}* {001,011,100,201}* {001,011,100,210}*	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$, $c_m = b_m0$	
	{001,011,102,120}* {001,012,101,110}* {001,011,021,120}* {001,011,101,120}* {001,011,110,120}* {001,011,120,201}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{001,011,120,210} {001,012,021,110}* {001,012,102,110}* {001,012,110,120}* {001,012,110,201}* {001,012,110,210	$a_1 \rightsquigarrow a_2, 01$; $01 \rightsquigarrow a_2, a_3$; $a_m \rightsquigarrow a_{m+1}$, where $a_m = 0^m$	
	{001,012,100,101}* {001,012,021,100}* {001,012,100,102}* {001,012,100,120}* {001,012,100,201}* {001,012,100,210}*	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$, $c_m = b_m0$	
	{000,001,110,120} {000,001,021,110}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow b_{m-1}, b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_mm$	$x + 2x^2 + \frac{3x^3}{1-x}$
15	{000,001,101,120}* {000,001,100,120}* {000,001,102,120}* {000,001,120,201}* {000,001,120,210}*	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow b_0, b_1, a_2$; $a_m \rightsquigarrow c_{m-1}, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_mm$, $c_m = b_m(m-1)$	
	{000,001,021,102}* {000,001,021,101}* {000,001,021,100}* {000,001,021,210}* {000,001,021,201}*	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow b_0, b_1, a_2$; $a_m \rightsquigarrow c_{m-1}, b_m, a_{m+1}$; $b_m \rightsquigarrow c_m$, where $a_m = 01\cdots m$, $b_m = a_mm$, $c_m = b_m0$	$x + 2x^2 + 3x^3 + \frac{4x^4}{1-x}$
16	{000,010,011,021}	$0 \rightsquigarrow a_0, 01$; $01 \rightsquigarrow b_2$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$	
	{010,011,012,021}	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$, where $a_m = 0^m$, $b_m = a_m1$	
	{001,010,021,100}* {001,010,021,101}* {001,010,021,102}* {001,010,021,110}* {001,010,021,120}* {001,010,021,201}* {001,010,021,210}* {001,010,100,101}* {001,010,100,102}* {001,010,100,110}* {001,010,100,120}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{001,010,100,201}*		
	{001,010,100,210}*		
	{001,010,101,102}*		
	{001,010,101,110}*		
	{001,010,101,120}*		
	{001,010,101,201}*		
	{001,010,101,210}*		
	{001,010,102,110}*		
	{001,010,102,120}*		
	{001,010,102,201}*		
	{001,010,102,210}*		
	{001,010,110,120}*		
	{001,010,110,201}*		
	{001,010,110,210}*		
	{001,010,120,201}*		
	{001,010,120,210}*		
	{001,010,201,210}*		
	{001,011,021,102}*		
	{001,011,101,102}*		
	{001,011,102,110}*		
	{001,011,102,201}*		
	{001,011,102,210}*		
	{001,011,021,101}*		
	{001,011,021,110}*		
	{001,011,021,201}*		
	{001,011,021,210}*		
	{001,011,101,110}*		
	{001,011,101,201}*		
	{001,011,101,210}*		
	{001,011,110,201}*		
	{001,011,110,210}*		
	{001,011,201,210}*	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$	
	{001,012,021,101}*		
	{001,012,101,102}*		
	{001,012,101,120}*		
	{001,012,101,201}*		
	{001,012,101,210}*		
	{001,012,021,102}*		
	{001,012,021,120}*		
	{001,012,021,201}*		
	{001,012,021,210}*		
	{001,012,102,120}*		
	{001,012,102,201}*		
	{001,012,102,210}*		
	{001,012,120,201}*		
	{001,012,120,210}*		
	{001,012,201,210}*	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$	
	{000,001,102,110}*		
	{000,001,101,110}*		
	{000,001,100,110}*		
	{000,001,110,201}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{000,001,110,210}*	$a_m \rightsquigarrow$ $(b_{m-1})^m, b_m, a_{m+1}$, where $a_m = 01\cdots m$, $b_m = a_m m$	$\frac{x}{(1-x)^2}$
17	{000,010,011,102}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow b_{m+1}, c_{m+2}$; $c_m \rightsquigarrow d_m, c_{m+1}$, where $a_m = 01\cdots m$, $b_m = 0a_m$, $c_m = 0a_{m-2}m$, $d_m = c_m(m-1)$	
	{000,010,011,120}	$0 \rightsquigarrow c_0, 01$; $01 \rightsquigarrow 012$; $012 \rightsquigarrow a_2$; $a_m \rightsquigarrow a_{m+1}$; $b_2 \rightsquigarrow 012, a_2$; $b_m \rightsquigarrow b_m, a_{m+1}$; $c_m \rightsquigarrow c_{m+1}, b_{m+2}$, where $c_m = 001\cdots m$, $b_m = c_{m-2}m$, $a_m = a_{m-3}(m-1)m$	
	{000,001,102,210}* {000,001,101,210}* {000,001,100,210}* {000,001,201,210}*	$a_0 \rightsquigarrow b_0, a_1$; $a_1 \rightsquigarrow b_0, b_1, a_2$; $a_m \rightsquigarrow$ $(c_{m-1})^m, b_m, a_{m+1}$; $b_m \rightsquigarrow (c_m)^m$, where $a_m = 01\cdots m$, $b_m = a_m m$, $c_m = b_m 0$	$\frac{x(1+x^3)}{(1-x)^2}$
18	{000,010,011,100}* {000,010,011,101}* {000,010,011,110}* {000,010,011,201}* {000,010,011,210}*	$a_0 \rightsquigarrow c_0, a_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow a_m, b_{m+1}$; $c_m \rightsquigarrow c_{m+1}, b_{m+2}$, where $a_m = 01\cdots m$, $b_m = 0a_{m-1}m$, $c_m = 0a_m$	
	{010,011,012,210}	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow (b_{m,1})^{j-1}$	$\frac{x(1-x+x^3)}{(1-x)^3}$
19	{010,011,012,100}* {010,011,012,101}* {010,011,012,102}* {010,011,012,110}* {010,011,012,120}* {010,011,012,201}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow b_{m,1}, \ldots, b_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{000,001,100,102}* {000,001,101,102}* {000,001,100,101}* {000,001,102,201}* {000,001,101,201}* {000,001,100,201}*	$a_0 \rightsquigarrow c_0, a_1$; $a_m \rightsquigarrow$ $b_{m-1,0}, \ldots, b_{m-1,m-2}$, c_{m-1}, c_m, a_{m+1}; $b_{m,j} \rightsquigarrow b_{m,0}, \ldots, b_{m,j-1}$; $c_m \rightsquigarrow b_{m,0}, \ldots, b_{m,m-1}$, where $a_m = 01\cdots m$, $c_m = a_m m$, $b_{m,j} = c_m j$	$\frac{x(1+x)}{1-x-x^2}$
27	{000,010,101,120} {000,010,110,120}		Following from Section 4.5 in [8]

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
40	{000,012,021,101}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 001$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 3x^4$
	{000,012,021,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 001$; $01 \rightsquigarrow 010, 011$; $001 \rightsquigarrow 0011$; $010 \rightsquigarrow 011$	
42	{000,012,100,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 011$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 011, 0011$; $010 \rightsquigarrow 011$	$x + 2x^2 + 4x^3 + 4x^4$
	{000,012,021,100} {000,012,021,102}* {000,012,021,120}* {000,012,021,201}* {000,012,021,210}*	$0 \rightsquigarrow 00, 00$; $00 \rightsquigarrow 001, 001$; $001 \rightsquigarrow 0011$	
43	{000,012,100,101}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 0021, 0022$; $0022 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 4x^4 + x^5$
	{000,012,102,110}* {000,012,110,120}* {000,012,110,201}* {000,012,110,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 011$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 001, 0011$; $010 \rightsquigarrow 011$	
44	{000,012,101,102}* {000,012,101,120}* {000,012,101,201}* {000,012,101,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 001, 0022$; $0022 \rightsquigarrow 00221$; $00221 \rightsquigarrow 002211$	$x + 2x^2 + 4x^3 + 4x^4 + 2x^5 + x^6$
45	{000,012,100,102}* {000,012,100,120}* {000,012,100,201}* {000,012,100,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 0021, 0022$; $010 \rightsquigarrow 0021$; $0022 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 5x^4 + x^5$
46	{000,012,102,120}* {000,012,102,201}* {000,012,102,210}* {000,012,120,201}* {000,012,120,210}* {000,012,201,210}*	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 001, 0022$; $010 \rightsquigarrow 0101$; $0022 \rightsquigarrow 00221$; $00221 \rightsquigarrow 002211$	$x + 2x^2 + 4x^3 + 5x^4 + 2x^5 + x^6$

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
47	{000,011,102,120}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 010, 012$; $00 \rightsquigarrow 01, b_2$; $012 \rightsquigarrow d_3$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $b_m \rightsquigarrow c_m, d_{m+1}$; $d_m \rightsquigarrow d_{m+1}$, where $a_m = 001 \cdots m$, $b_m = a_{m-2}m$, $c_m = a_{m-2}m(m-1)$, $d_m = a_{m-3}(m-1)m$	
	{000,011,021,102}	$a_m \rightsquigarrow b_m, a_{m+1}$; $b_1 \rightsquigarrow d_1, c_2$; $c_m \rightsquigarrow c_{m+1}$; $d_m \rightsquigarrow d_{m+1}, c_{m+2}$, where $a_m = 01 \cdots m$, $b_m = a_{m-1}0$, $c_m = 0a_{m-2}m$, $d_m = 0a_m$	
	{000,011,021,120}	$0 \rightsquigarrow a_0, 01$; $01 \rightsquigarrow 010, b_2$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $010 \rightsquigarrow b_2$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 001 \cdots m$, $b_m = a_{m-2}m$	
	{001,100,110,120}	$a_1 \rightsquigarrow a_2, c_1$; $a_m \rightsquigarrow a_{m+1}$; $c_m \rightsquigarrow b_m, a_{m+2}, c_{m+1}$, where $a_m = 0^m$, $c_m = 01 \cdots m$, $b_m = a_m(m-1)$	
	{001,021,100,120}	$a_1 \rightsquigarrow a_2, d_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow c_m, b_{m+1}$; $d_1 \rightsquigarrow c_1, b_2, a_2$; $d_m \rightsquigarrow a_{m+2}, d_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$, $c_m = b_m0$, $d_m = 01 \cdots m$	
	{001,021,110,120}	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_1 \rightsquigarrow a_2, a_3, b_2$, $b_m \rightsquigarrow a_{m+2}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01 \cdots m$	
	{001,021,100,110}	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow c_m, a_{m+2}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01 \cdots m$, $c_m = b_m0$	$\frac{x(1+x^2-x^3)}{(1-x)^2}$
48	{000,011,100,102}* {000,011,101,102}* {000,011,102,110}* {000,011,102,201}* {000,011,102,210}* {000,011,100,120}* {000,011,101,120}* {000,011,110,120}* {000,011,120,201}* {000,011,120,210}* {000,011,021,100}* {000,011,021,101}* {000,011,021,110}* {000,011,021,201}*	$0 \rightsquigarrow a_0, 01$; $01 \rightsquigarrow 010, b_2$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 01 \cdots m$, $b_m = a_{m-2}m$, $c_m = b_m(m-1)$	

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	$\{000,011,021,210\}^*$	$0 \rightsquigarrow a_0, 01$; $01 \rightsquigarrow 010, 012$; $010 \rightsquigarrow 012$; $012 \rightsquigarrow c_3$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $b_m \rightsquigarrow c_m, c_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 001\cdots m$, $b_m = a_{m-2}m$, $c_m = b_{m-1}m$	
	$\{011,012,021,100\}$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow c_m$, where $a_m = 0^m$, $b_m = a_m1$, $c_m = b_m0$	
	$\{001,100,102,120\}^*$ $\{001,100,101,120\}^*$ $\{001,021,100,102\}^*$ $\{001,100,120,201\}^*$ $\{001,100,120,210\}$ $\{001,021,100,101\}^*$ $\{001,021,100,210\}^*$ $\{001,021,100,201\}^*$	$a_1 \rightsquigarrow a_2, b_1$; $b_1 \rightsquigarrow c_1, b_2, d_2$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow c_m, b_{m+1}$; $d_m \rightsquigarrow c_m, b_{m+1}, d_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$, $c_m = b_m0$, $d_m = 01\cdots m$	
	$\{001,102,110,120\}^*$ $\{001,101,110,120\}^*$ $\{001,021,102,110\}^*$ $\{001,110,120,201\}^*$ $\{001,110,120,210\}^*$ $\{001,021,101,110\}^*$ $\{001,021,110,210\}^*$ $\{001,021,110,201\}^*$	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow a_{m+1}, a_{m+2}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$	
	$\{001,100,102,110\}^*$ $\{001,100,101,110\}^*$ $\{001,100,110,201\}^*$ $\{001,100,110,210$	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow (c_m)^m, a_{m+2}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$, $c_m = b_m0$	
	$\{001,021,101,120\}^*$ $\{001,021,102,120\}^*$ $\{001,021,120,201\}^*$ $\{001,021,120,210\}^*$	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_1 \rightsquigarrow a_2, b_2, b_2$; $b_m \rightsquigarrow a_{m+1}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$	$\frac{x(1+x^2)}{(1-x)^2}$
49	$\{000,011,100,101\}^*$ $\{000,011,100,110\}^*$ $\{000,011,100,201\}^*$ $\{000,011,100,210\}^*$ $\{000,011,101,110\}^*$ $\{000,011,101,201\}^*$ $\{000,011,101,210\}^*$ $\{000,011,110,201\}^*$ $\{000,011,110,210\}^*$		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	$\{000,011,201,210\}^*$	$0 \rightsquigarrow a_0, 01$; $01 \rightsquigarrow 010, b_2$; $010 \rightsquigarrow c_2$; $a_m \rightsquigarrow a_{m+1}, b_{m+2}$; $b_m \rightsquigarrow c_m, b_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 001\cdots m$, $b_m = a_{m-2}m$, $c_m = b_m(m-1)$	
	$\{010,012,021,100\}^*$ $\{010,012,021,101\}^*$ $\{010,012,021,102\}^*$ $\{010,012,021,110\}^*$ $\{010,012,021,120\}^*$ $\{010,012,021,201\}^*$ $\{010,012,021,210\}^*$ $\{011,012,021,101\}^*$ $\{011,012,021,102\}^*$ $\{011,012,021,110\}^*$ $\{011,012,021,120\}^*$ $\{011,012,021,201\}^*$ $\{011,012,021,210\}^*$	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow b_{m+1}$, where $a_m = 0^m$, $b_m = a_m 1$	
	$\{001,021,201,210\}^*$ $\{001,101,102,120\}^*$ $\{001,101,120,201\}^*$ $\{001,101,120,210\}^*$ $\{001,102,120,201\}^*$ $\{001,102,120,210\}^*$ $\{001,021,101,102\}^*$ $\{001,021,102,210\}^*$ $\{001,021,102,201\}^*$ $\{001,021,101,201\}^*$ $\{001,021,101,210\}^*$ $\{001,120,201,210\}^*$	$a_1 \rightsquigarrow a_2, c_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow a_{m+1}, b_{m+1}$; $c_m \rightsquigarrow a_{m+1}, b_{m+1}, c_{m+1}$, where $a_m = 0^m$, $b_m = 01^m$, $c_m = 01\cdots m$	
	$\{001,101,102,110\}^*$ $\{001,102,110,201\}^*$ $\{001,102,110,210\}^*$ $\{001,101,110,201\}^*$ $\{001,101,110,210\}^*$ $\{001,110,201,210\}^*$	$a_1 \rightsquigarrow a_2, b_1$; $a_m \rightsquigarrow a_{m+1}$; $b_m \rightsquigarrow$ $(a_{m+1})^m, a_{m+2}, b_{m+1}$, where $a_m = 0^m$, $b_m = 01\cdots m$	
	$\{001,100,102,210\}^*$ $\{001,100,101,210\}^*$ $\{001,100,201,210\}^*$	$a_1 \rightsquigarrow a_2, d_{1,1}$; $a_m \rightsquigarrow a_{m+1}$; $d_{m,1} \rightsquigarrow$ $(c_m)^m, d_{m,2}, d_{m+1,1}$; $d_{m,j} \rightsquigarrow$ $(c_{m+j-1})^m, d_{m,j+1}$, where $a_m = 0^m$, $c_m = 01^m0$, $d_{j,i} = 01\cdots(j-1)j^i$	$\frac{x(1-x+x^2)}{(1-x)^3}$

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
51	{011,012,100,201}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow c_m, b_{m,1}, \ldots, b_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 10$	$\frac{x}{(1-x)(1-x-x^2)}$
	{001,100,101,102}* {001,100,102,201}* {001,100,101,201}*	$a_1 \rightsquigarrow a_2, c_{1,1}$; $a_m \rightsquigarrow a_{m+1}$; $c_{m,1} \rightsquigarrow d_{1,m}, \ldots, d_{m,1}, c_{m,2}, c_{m+1,1}$; $c_{m,j} \rightsquigarrow d_{1,m+j-1}, \ldots, d_{m,j}, c_{m,j+1}$; $d_{m,j} \rightsquigarrow d_{1,m+j-1}, \ldots, d_{m-1,j+1}$, where $a_m = 0^m$, $c_{m,j} = 01\cdots(m-1)m^j$, $d_{m,j} = c_{m,j}(m-1)$	
52	{011,012,100,101}* {011,012,100,102}* {011,012,100,110}* {011,012,100,120}*	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow c_{m,1}$; $b_{m,2} \rightsquigarrow (b_{m,1})^2$; $b_{m,j} \rightsquigarrow c_{m,j}, b_{m,1}, \ldots, b_{m,j-1}$; $c_{m,j} \rightsquigarrow c_{m+1,1}, b_{m+1,1}, \ldots, b_{m+1,j-2}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = b_{m,j}0$	$\frac{x(1-x^2-x^3)}{(1-x-x^2)^2}$
55	{010,012,100,210} {011,012,101,210}* {011,012,102,210}* {011,012,110,210}* {011,012,120,210}*	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow (c_m)^{j-1}, b_{m+1,j}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 21$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
	{010,012,110,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow (b_{m,1})^{j-1}, b_{m+1,1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{001,101,102,210}* {001,102,201,210}* {001,101,201,210}*	$a_{m,1} \rightsquigarrow (a_{0,m+1})^m, a_{m,2}, a_{m+1,1}$; $a_{m,j} \rightsquigarrow (a_{0,m+1})^m, a_{m+1,j+1}$, where $a_{m,j} = 01\cdots(m-1)m^j$	
56	{010,012,100,101}* {010,012,100,102}* {010,012,100,120}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{010,012,100,201}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $c_{m,2}, \ldots, c_{m,j}, b_{m+1,j}$; $c_{m,j} \rightsquigarrow c_{m,2}, \ldots, c_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = b_{m,j}(j-1)$	
	{010,012,101,110} {010,012,102,110}* {010,012,110,120}* {010,012,110,201}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m,1}, \ldots, b_{m,j-1}, b_{m+1,1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{011,012,101,201}* {011,012,102,201}* {011,012,110,201}* {011,012,120,201}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m,1}, b_{m,1}, \ldots, b_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	$\frac{x(1-x+x^3)}{(1-x-x^2)(1-x)^2}$
57	{010,012,101,210}* {010,012,102,210}* {010,012,120,210}* {010,012,201,210}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow (b_{m,1})^{j-1}, b_{m+1,j}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	$\frac{x(1-3x+4x^2-2x^3+x^4)}{(1-x)^5}$
58	{010,011,021,100}* {010,011,021,101}* {010,011,021,102}* {010,011,021,110}* {010,011,021,120}* {010,011,021,201}* {010,011,021,210}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m+1,1}, \ldots, b_{m+1,m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{010,012,101,102}* {010,012,101,120}* {010,012,101,201}* {010,012,102,120}* {010,012,102,201}* {010,012,120,201}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m,1}, \ldots, b_{m,j}, b_{m+1,j}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{011,012,101,102}* {011,012,101,110}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{011,012,101,120}* {011,012,102,110}* {011,012,102,120}* {011,012,110,120}*	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots,$ $b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m,j}, b_{m,1}, \ldots, b_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	$\frac{x}{1-2x}$
	{001,101,102,201}*	$a_{m,1} \rightsquigarrow$ $a_{0,m+1}, \ldots, a_{m-1,2}, a_{m,2}, a_{m+1,1}$; $a_{m,j} \rightsquigarrow$ $a_{0,m+j}, a_{1,m+j-1}, \ldots, a_{m-1,j+1}$, $a_{m,j+1}$, where $a_m = 01\cdots(j-1)j^m$	
61	{010,011,102,201} {010,011,120,201}		Theorem 6
63	{010,011,100,102}* {010,011,101,102}* {010,011,102,110}*	$\overset{w}{\sim}$010,011,102	
64	{000,010,021,100}* {000,010,021,101}* {000,010,021,102}* {000,010,021,110}* {000,010,021,120}* {000,010,021,201}* {000,010,021,210}*	$\overset{w}{\sim}$000,010,021	
65	{010,011,100,120}* {010,011,101,120}* {010,011,110,120}*	$\overset{w}{\sim}$010,011,120	
67	{010,011,100,201}* {010,011,101,201}* {010,011,110,201}*	$\overset{w}{\sim}$010,011,201	
68	{010,011,100,210}* {010,011,101,210}* {010,011,110,210}*	$\overset{w}{\sim}$010,011,210	
69	{010,011,100,101}* {010,011,100,110}* {010,011,101,110}*	$\overset{w}{\sim}$010,011	
70	{012,021,100,101}	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow c_m, b_{m+1}$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 0$	$\frac{x(1-x+2x^2)}{(1-x)^3}$
	{012,021,100,110} {012,021,101,110}	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow (c_{m-1})^2$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 1$	
73	{012,021,100,102}* {012,021,100,120}* {012,021,100,201}* {012,021,100,210}* {012,021,101,102}* {012,021,101,120}* {012,021,101,201}* {012,021,101,210}*	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow c_m, b_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 0$	
	{012,021,102,110}* {012,021,110,120}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{012,021,110,201}* {012,021,110,210}*	$a_m \rightsquigarrow a_{m+1}, (b_m)^m$; $b_m \rightsquigarrow c_m, b_{m+1}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_m = a_m1$, $c_m = b_m1$	$\frac{x(1-2x+3x^2-x^3)}{(1-x)^4}$
76	{012,100,102,110}* {012,100,110,120}*	$\overset{w}{\sim}$012,100,110	
79	{012,100,101,201}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow c_{m,1}, \ldots, c_{m,j}, b_{m+1,j}$; $c_{m,j} \rightsquigarrow c_{m,1}, \ldots, c_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_mj$, $c_{m,j} = b_{m,j}(j-1)$	$\frac{x(1-x+x^2+x^3)}{(1-x)^2(1-x-x^2)}$
	{012,101,110,201}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{1,1} \rightsquigarrow 010, c_1$; $010 \rightsquigarrow c_1$; $b_{m,j} \rightsquigarrow c_{m-1}, b_{m,1}, \ldots, b_{m,j-1}, c_m$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_{m,j} = a_mj$, $c_m = a_m11$	
82	{012,100,101,102}* {012,100,101,120}*	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow c_{m,1}, b_{m+1,1}$; $b_{m,j} \rightsquigarrow c_{m,j}, c_{m,2}, d_{m,3}, \ldots, d_{m,j}, b_{m+1,j}$; $c_{m,j} \rightsquigarrow c_{m,1}, \ldots, c_{m,j-1}$; $d_{m,j} \rightsquigarrow (c_{m,2})^2, d_{m,3}, \ldots, d_{m,j-1}$, where $a_m = 0^m$, $b_{m,j} = a_mj$, $c_{m,j} = b_{m,j}0$, $d_{m,j} = b_{m,j}(j-1)$	$\frac{x(1-2x+x^2+x^3-x^4-x^5)}{(1-x)^2(1-x-x^2)^2}$
83	{011,021,100,102}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow c_m, b_{m+1,j+1}, \ldots, b_{m+1,m+1}$, where $a_m = 0^m$, $b_{m,j} = a_mj$	
	{011,021,100,120}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow c_{m,0}, c_{m,2}, \ldots, c_{m,m+1}$; $b_{m,j} \rightsquigarrow c_{m-1,j}, c_{m,j+1}, \ldots, c_{m,m+1}$; $c_{m,0} \rightsquigarrow c_{m,2}, \ldots, c_{m,m+1}$; $c_{m,j} \rightsquigarrow c_{m+1,j+1}, \ldots, c_{m+1,m+2}$, where $a_m = 0^m$, $b_{m,j} = a_mj$, $c_{m,j} = a_m1j$	
	{012,101,102,110}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{012,101,110,120}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow c_{m-1}, c_m$; $b_{m,j} \rightsquigarrow$ $d_{m,j}, b_{m,1}, \ldots, b_{m,j-1}, c_m$; $c_m \rightsquigarrow c_{m+1}$; $d_{m,j} \rightsquigarrow$ $d_{m+1,j}, b_{m+1,1}, \ldots, b_{m+1,j}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 11$, $d_{m,j} = b_{m,j}0$	$\frac{x(1-x+x^2)}{(1-x)(1-2x)}$
86	{000,021,101,120} {000,021,101,102} {000,021,110,120}		See Subsection 4.1
87	{012,100,201,210}	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $c_m, (d_m)^{j-1}, b_{m+1,j}$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 10$, $d_m = a_m 21$	
	{012,110,201,210}	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m,1}, (c_{m-1})^{j-1}, c_m$; $c_m \rightsquigarrow c_{m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 11$	$\frac{x(1-2x+3x^2)}{(1-x)^4}$
88	{000,021,102,201}* {000,021,102,210}* {000,021,100,102}*	$\overset{w}{\approx}$000,021,102	
89	{012,100,102,210}* {012,100,120,210}* {012,101,201,210} {012,102,110,210}* {012,110,120,210}*		See Subsection 4.2
90	{012,100,102,201}* {012,100,120,201}*	$\overset{w}{\approx}$012,100,201	
	{012,102,110,201}* {012,110,120,201}*	$\overset{w}{\approx}$012,110,201	See Section 4.2 in [8]
92	{012,101,102,210}* {012,101,120,210}*	$\overset{w}{\approx}$012,101,210	
93	{011,102,120,201} {011,102,120,210}		Theorem 7
	{011,021,101,102}* {011,021,102,110}* {011,021,102,201}* {011,021,102,210}*	$\overset{w}{\approx}$011,021,102	
	{012,101,102,201}* {012,101,120,201}*	$\overset{w}{\approx}$012,101,201	
	{012,102,110,120}*	$\overset{w}{\approx}$012,110	
	{012,021,102,120}* {012,021,102,201}* {012,021,102,210}* {012,021,120,201}* {012,021,120,210}* {012,021,201,210}*	$\overset{w}{\approx}$012,021	See Class 45 in Table 2 and Section 3.2 in [8]
94	{000,021,120,201}* {000,021,120,210}* {000,021,100,120}*	$\overset{w}{\approx}$000,021120	
95	{011,101,102,120}* {011,102,110,120}*	$\overset{w}{\approx}$011,102,120	

Continuation of Table 2

Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	$\{011,021,100,101\}^*$		
	$\{011,021,100,110\}^*$		
	$\{011,021,100,201\}^*$		
	$\{011,021,100,210\}^*$	$\overset{w}{\sim}011,021,100$	
	$\{011,021,101,120\}^*$		
	$\{011,021,110,120\}^*$		
	$\{011,021,120,201\}^*$		
	$\{011,021,120,210\}^*$	$\overset{w}{\sim}011,021,120$	See Section 4.3 in [8]
97	$\{000,021,110,210\}^*$		
	$\{000,021,110,201\}^*$		
	$\{000,021,100,110\}^*$	$\overset{w}{\sim}011,021,110$	
99	$\{011,100,102,201\}$		
	$\{011,100,120,201\}$		Theorem 5
101	$\{011,100,101,102\}^*$		
	$\{011,100,102,110\}^*$	$\overset{w}{\sim}011,100,102$	
102	$\{011,100,101,120\}^*$		
	$\{011,100,110,120\}^*$	$\overset{w}{\sim}011,100,120$	
105	$\{000,021,101,201\}^*$		
	$\{000,021,101,210\}^*$		
	$\{000,021,100,101\}^*$	$\overset{w}{\sim}000,021,101$	
109	$\{012,102,201,210\}^*$		
	$\{012,120,201,210\}^*$	$\overset{w}{\sim}012,201,210$	
113	$\{012,102,120,201\}^*$	$\overset{w}{\sim}012,201$	
	$\{012,102,120,210\}^*$	$\overset{w}{\sim}012,210$	See Section 3.3 in [8]
114	$\{011,101,102,210\}^*$		
	$\{011,102,110,210\}^*$	$\overset{w}{\sim}011,102,210$	
115	$\{011,101,102,201\}^*$		
	$\{011,102,110,201\}^*$	$\overset{w}{\sim}011,102,201$	
119	$\{000,021,100,201\}^*$		
	$\{000,021,100,210\}^*$		
	$\{000,021,201,210\}^*$	$\overset{w}{\sim}000,021$	
120	$\{011,101,120,201\}^*$		
	$\{011,110,120,201\}^*$	$\overset{w}{\sim}011,120,201$	
121	$\{011,100,201,210\}$		
	$\{011,101,102,110\}^*$		See Section 4.3
123	$\{011,101,120,210\}^*$		
	$\{011,110,120,210\}^*$	$\overset{w}{\sim}011,120,210$	
125	$\{011,100,101,201\}^*$		
	$\{011,100,110,201\}^*$	$\overset{w}{\sim}011,100,201$	
	$\{011,021,101,110\}^*$		
	$\{011,021,101,201\}^*$		
	$\{011,021,101,210\}^*$		
	$\{011,021,110,201\}^*$		
	$\{011,021,110,210\}^*$		
	$\{011,021,201,210\}^*$	$\overset{w}{\sim}011,021$	See Section 4.4 in [8]
127	$\{011,100,101,210\}^*$		
	$\{011,100,110,210\}^*$	$\overset{w}{\sim}011,100,210$	
145	$\{011,101,201,210\}^*$		
	$\{011,110,201,210\}^*$	$\overset{w}{\sim}011,201,210$	
149	$\{010,021,100,101\}^*$		
	$\{010,021,100,102\}^*$		
	$\{010,021,100,110\}^*$		
	$\{010,021,100,120\}^*$		
	$\{010,021,100,201\}^*$		
	$\{010,021,100,210\}^*$		
	$\{010,021,101,102\}^*$		
	$\{010,021,101,110\}^*$		
	$\{010,021,101,120\}^*$		
	$\{010,021,101,201\}^*$		
	$\{010,021,101,210\}^*$		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{010,021,102,110}* {010,021,102,120}* {010,021,102,201}* {010,021,102,210}* {010,021,110,120}* {010,021,110,201}* {010,021,110,210}* {010,021,120,201}* {010,021,120,210}* {010,021,201,210}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $b_{m+1,j}, \ldots, b_{m+1,m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{011,101,110,201}*	$\overset{w}{\sim}$ 011, 201	$\frac{1-2x-\sqrt{1-4x}}{2x}$
162	{010,100,101,120} {010,100,110,120}		Theorem 8
173	{010,100,120,201} {010,110,120,201}		Followed from Subsection 4.7 in [5]
174	{010,100,110,201} {010,101,110,201}		Followed from Subsection 4.6 in [8]
178	{010,101,120,210} {010,110,120,210}		Followed from Subsection 4.5 in [8]
185	{010,100,201,210} {010,101,201,210}		Followed from Subsection 4.6 in [8]
186	{021,100,102,120} {021,101,102,120} {021,102,110,120}		Followed from Subsection 4.7 in [8]
190	{021,100,101,120} {021,101,110,120} {021,100,110,120}	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $c_{m-1,j}, c_{m,j}, \ldots, c_{m,m+1}$; $c_{m,j} \rightsquigarrow$ $c_{m+1,j}, \ldots, c_{m+1,m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$	$\frac{(1+x)(1-\sqrt{1-4x})}{2x} - \frac{x}{1-x} - 1$
	{021,100,101,102}	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $c_m, b_{m+1,j}, \ldots, b_{m+1,m+1}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 10$	
191	{021,102,110,201}* {021,102,110,210}*	$\overset{w}{\sim}$ 021, 102, 110	
192	{021,102,120,201}* {021,102,120,210}*	$\overset{w}{\sim}$ 021, 102, 120	
193	{021,100,102,201}* {021,100,102,210}*	$\overset{w}{\sim}$ 021, 100, 102	
194	{021,101,102,201}* {021,101,102,210}*	$\overset{w}{\sim}$ 021, 101, 102	
195	{021,100,110,201}* {021,100,110,210}*	$\overset{w}{\sim}$ 021, 100, 110	
196	{021,101,120,201}* {021,101,120,210}* {021,100,120,201}*		

Continuation of Table 2			
Class	B quadruple	Rules of $\mathcal{T}'(B)$	$G'_B(x)$/Reference
	{021,100,120,210}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow$ $c_{m,0}, b_{m+1,1}, c_{m,2}, \ldots, c_{m,m+1}$; $b_{m,j} \rightsquigarrow$ $c_{m-1,j}, b_{m+1,j}, c_{m,j+1}, \ldots, c_{m,m+1}$; $c_{m,0} \rightsquigarrow$ $c_{m+1,0}, c_{m+1,2}, \ldots, c_{m+1,m+2}$; $c_{m,j} \rightsquigarrow$ $c_{m+1,j}, \ldots, c_{m+1,m+2}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$	
	{021,110,120,201}* {021,110,120,210}*	$a_m \rightsquigarrow$ $a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,1} \rightsquigarrow c_{m,0}, \ldots, c_{m,m+1}$; $b_{m,j} \rightsquigarrow$ $c_{m-1,j-1}, c_{m,j}, \ldots, c_{m,m+1}$; $c_{m,0} \rightsquigarrow$ $c_{m+1,0}, c_{m,1}, \ldots, c_{m,m+1}$; $c_{m,j} \rightsquigarrow$ $c_{m+1,j}, \ldots, c_{m+1,m+2}$, where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$	$\frac{2-7x+4x^2+(3x-2)\sqrt{1-4x}}{2x(1-x)}$
197	{021,101,110,201}* {021,101,110,210}*	$\overset{w}{\sim}$ 021, 101, 110	
201	{021,100,101,201}* {021,100,101,210}*	$\overset{w}{\sim}$ 021, 100, 101	
205	{100,102,120,201} {102,110,120,201}		Theorem 9
206	{100,102,120,210} {101,102,120,201} {101,102,120,210}		
	{102,110,120,210}		Theorem 10
211	{101,102,110,201} {021,120,201,210}*		Subsection 4.5
214	{100,101,120,201} {101,110,120,201}		Following from Lemmas 13 and 14 in [5]
215	{100,101,110,201} {021,101,201,210}*		Open
219	{100,101,102,201} {100,110,120,201}		Theorem 11
226	{101,110,201,210} {100,110,201,210}	Subsection 4.6	
227	{100,120,201,210} {110,120,201,210}		Followed by Section 4.8 in [8]
End of Table 2			

Table 2: Rules of generating trees for weak ascent sequences avoiding a quadruple of length-3 patterns.

Proposition 1. (*i*) *The number of weakly increasing weak ascent sequences of length n is the Catalan number C_n; (ii) The number of weakly increasing weak ascent sequences of length n that avoid the pattern 000 is the Motzkin number M_n.*

Proof. (i) A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin and stays weakly above the x-axis. Dyck n-paths are counted by C_n. Define a map ϕ from a Dyck n-path P to a length-n nonnegative integer sequence $\{(w_i)_{i=1}^n\}$ by letting w_i be the distance from the ith upstep of P northwest to the diagonal line $y = x$. Then ϕ is a bijection from Dyck n-paths to weakly increasing weak ascent sequences of length n.

(ii) The restriction of ϕ to Dyck n-paths with no 3 (contiguous) upsteps UUU is a bijection to weakly increasing weak ascent sequences of length n that avoid the pattern 000. The comments on A001006 in OEIS give a short proof that UUU-free Dyck paths are counted by the Motzkin numbers M_n. □

4.1. Class 86

Let $P = \{000, 021, 101, 120\}$, $Q = \{000, 021, 101, 102\}$, and $R = \{000, 021, 110, 120\}$.

Theorem 4. *We have*

$$|WA_n(P)| = |WA_n(Q)| = |WA_n(R)| = M_n + M_{n-2}, \quad \textit{for } n \geq 3.$$

Proof. First note that a weakly increasing weak ascent sequence that avoids 000 also avoids all of the other patterns in question and so is a P, Q, and R avoider.In each of cases P, Q, R, we give a bijection from the length-n avoiders that are *not* weakly increasing to the length-$(n-2)$ weakly increasing weak ascent sequences that avoid 000. The theorem then follows from Proposition 1(ii).

A weak ascent sequence that avoids 021 can only have 0 as a descent bottom and if it also avoids 000, then 0 can occur only once as a descent bottom. So all of our avoiders have a unique descent bottom 0.

A case P avoider $w = (w_i)_{i=1}^n$ of length $n \geq 3$ that is not weakly increasing must begin 010 or 0110 for if it begins 012 or 0112, the descent bottom would end a 120; also, all later entries must be at least 2 or a 000 or 101 would be present. The map ψ_P defined by

$$\psi_P(w) = \begin{cases} 0\,(w_i - 1)_{i=4}^n & \text{if } w_3 = 0, \\ 0\,0\,(w_i - 1)_{i=5}^n & \text{if } w_3 = 1, \end{cases}$$

is the desired bijection.

A case Q avoider $w = (w_i)_{i=1}^n$ of length $n \geq 3$ that is not weakly increasing must end with a descent to 0, say $a0$ because $a0b$ would be a 102 if $b > a$ and a 101 if $b = a$ while if $0 < b < a$, then $0ab$ would be a 021, and $b = 0$ would produce a 000. The interior entries of w are all positive to avoid 000. Here the map ψ_Q defined by $\psi_Q(w) = (w_i - 1)_{i=2}^{n-1}$ is the desired bijection.

A case R avoider $w = (w_i)_{i=1}^n$ of length $n \geq 3$ that is not weakly increasing must begin 010. For otherwise, there are at least two entries between the two 0's and they must be distinct or a 110 is present. Since they are increasing, they must form an initial segment of the positive integers because w is a weak ascent sequence, and so a forbidden 120 would be present. Here the map ψ_R defined by $\psi_R(w) = 0\,(w_i - 1)_{i=4}^n$ is the desired bijection. □

4.2. Class 89

By Theorem 3, we have

$$\{012, 100, 102, 210\} \overset{w}{\sim} \{012, 100, 120, 210\} \overset{w}{\sim} \{012, 100, 210\}$$

and

$$\{012, 102, 110, 210\} \overset{w}{\sim} \{012, 110, 120, 210\} \overset{w}{\sim} \{012, 110, 210\}.$$

Thus, by Section 4.1 in [8], we obtain

$$\begin{aligned}\sum_{n\geq 0} |WA_n(012, 100, 102, 210)|x^n &= \sum_{n\geq 0} |WA_n(012, 100, 120, 210)|x^n \\ = \sum_{n\geq 0} |WA_n(012, 102, 110, 210)|x^n &= \sum_{n\geq 0} |WA_n(012, 110, 120, 210)|x^n \\ = \frac{x(x^4 - 3x^3 + 5x^2 - 3x + 1)}{(1-x)^5} &= x + 2x^2 + 5x^3 + 12x^4 + \dots .\end{aligned}$$

Note that this generating function corrects the generating function in Section 4.1 in [8] which begins $2x + 2x^2 + 5x^3 + 12x^4 + \dots$ (the difference between the two is only an x summand). Thus, it remains to find the generating function $F(x) = \sum_{n\geq 0} |WA_n(012, 101, 201, 210)|x^n$. By our strategy, we find the the generating tree $\mathcal{T}'(012, 101, 201, 210)$ satisfies the following rules:

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow (c_m)^j, b_{m+1,j}, \\ c_m &\rightsquigarrow c_{m+1}, \end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_m = a_m 10$.

Define $A_m(x)$ (resp. $B_{m,j}(x)$, $C_m(x)$) to be the generating function for the number nodes at level n in the subtree $\mathcal{T}'(012, 101, 201, 210; a_m)$ (resp. $\mathcal{T}'(012, 101, 201, 210; b_{m,j})$, $\mathcal{T}'(012, 101, 201, 210; c_m)$), where the root stays at level 1. The rules lead to the following recurrence relations:

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + x\sum_{j=1}^{m} B_{m,j}(x), \\ B_{m,j}(x) &= x + jxC_m(x) + xB_{m+1,j}(x), \\ C_m(x) &= x + xC_{m+1}(x). \end{aligned}$$

Hence, by induction on m, we have that $C_m(x) = \frac{x}{1-x}$. Thus, $B_{m,j}(x) = x + \frac{jx^2}{1-x} + xB_{m+1,j}(x)$. Define $B(x, u, v) = \sum_{m\geq 1} \sum_{j=1}^{m} B_{m,j}(x)u^{m-j}v^{m-1}$. Then, $B(x, u, v)$ satisfies

$$B(x, u, v) = \frac{x}{(1-v)(1-uv)} + \frac{x^2}{(1-x)(1-v)^2(1-uv)} + \frac{x}{uv}(B(x, u, v) - B(x, 0, v)).$$

Taking $u = x/v$, we obtain

$$B(x, 0, v) = \frac{x(1 - v + vx)}{(1 - x)^2(1 - v)^2},$$

which leads to,

$$B(x, u, v) = \frac{x(1 - v + vx)}{(1 - x)^2(1 - v)^2(1 - uv)}.$$

Define $A(x, v) = \sum_{m\geq 1} A_m(x)v^{m-1}$. By the recurrence for $A_m(x)$, we have

$$A(x, v) = \frac{x}{1 - v} + \frac{x}{v}(A(x, v) - A(x, 0)) + xB(x, 1, v).$$

Taking $v = x$, we obtain

$$A(x, 0) = \frac{x(x^4 - 3x^3 + 5x^2 - 3x + 1)}{(1 - x)^5}.$$

4.3. Class 121

Note that $\{011, 101, 102, 110\} \overset{w}{\sim} \{011, 102\}$. By Section 3.4 in [8], we have

$$\sum_{n\geq 1} |WA_n(\{011, 102\})|x^n = \frac{x(1 - x)}{1 - 3x + x^2}.$$

Thus, the aim of this section is to show

$$\sum_{n\geq 1} |WA_n(\{011, 100, 201, 210\})|x^n = \frac{x(1 - x)}{1 - 3x + x^2}. \tag{1}$$

A *descent pair* in a sequence w is a pair of consecutive entries w_i, w_{i+1} with $w_i > w_{i+1}$ and w_i is a *descent top.* For a weak ascent sequence w, we say w_k is a *big* entry if $k \geq 2$ and w_k is equal to 1 plus the number of the weak ascents in $(w_i)_{i=1}^{k-1}$ (so $w_k \geq 1$ has its maximum allowed value). Let W_n denote $WAS_n(011, 100, 201, 210)$. A weak ascent sequence $(w_i)_{i=1}^{n}$ is in W_n if and only if its positive entries are distinct (to avoid 011) and, after a descent $w_i > w_{i+1}$, all later entries are greater than w_i (to avoid the other three triples). The following lemma is evident from this characterization and the definition of weak ascent sequence.

Lemma 1. *For $w \in W_n$ and $2 \leq k \leq n - 1$, if w_k is a big entry, then either w_{k+1} is also a big entry, or w_k, w_{k+1} is a descent pair and w_j is a big entry for $j \geq k + 2$.* □

Let $W_{n,k}$ denote the subset of W_n for which the last descent top occurs in position k (taken to be 0 if there are no descents). For example,

$$W_{4,3} = \{0010, 0020, 0021, 0120\}.$$

Set $u_n = |W_n|$ and let $u_{n,k} = |W_{n,k}|$. The first few values of $u_{n,k}$ are given in Table 3.

$n \setminus k$	0	1	2	3	4	5	6	7
1	1	0						
2	2	0						
3	4	0	1					
4	8	0	1	4				
5	16	0	1	5	12			
6	32	0	1	6	17	33		
7	64	0	1	7	23	50	88	
8	128	0	1	8	30	73	138	232

Table 3: Table of values of $u_{n,k}$

4.3.1. The Recurrence

We will show that $u_{n,k}$ satisfies the following recurrence:

$$\begin{aligned}
&u_{n,0} = 2^{n-1} && \text{for } n \geq 1,\\
&u_{n,1} = 0 && \text{for } n \geq 1,\\
&u_{n,2} = 1 && \text{for } n \geq 3,\\
&u_{n,k} = u_{n-1,k} + u_{n-1,k-1} && \text{for } 2 < k < n-1,\\
&u_{n,k} = u_{n,n-2} + u_{n-1,n-2} + n - 2 && \text{for } k = n-1\,.
\end{aligned}$$

This recurrence leads to the generating function

$$U(x,y) = \sum_{n\geq 1,\, k\geq 0} u_{n,k} x^n y^k = \frac{x}{1-2x} + \frac{x^3y^2}{(1-x-xy)(1-3xy+x^2y^2)}\,.$$

Note that $U(x,1) = \frac{x(1-x)}{1-3x+x^2}$, which completes the proof of 1.

To establish the recurrence, the case $k = 0$ is known. For $n \geq 1$, $W_{n,1}$ is clearly empty. For $n \geq 3$, $W_{n,2}$ consists of the singleton (0,1,0) when $n = 3$ and the singleton $(0, 1, 0, 2, 3, \ldots, n-2)$ when $n \geq 4$.

Next suppose $2 < k < n-1$. By Lemma 1, if $w \in W_{n,k}$ has a big entry, then w_n is a big entry. So the map "delete last entry" is a bijection from $\{w \in W_{n,k} : w \text{ contains a big entry}\}$ to $W_{n-1,k}$. For example, 00103245 $\to$ 0010324. To reverse the map, given $w \in W_{n-1,k}$, append to w the entry $1+\#$ weak ascents in w. Contrariwise, the map "delete first entry" is a bijection from $\{w \in W_{n,k} : w \text{ has no big entry}\}$ to $W_{n-1,k-1}$. For example, 0001243 $\to$ 001243. To reverse the map, given $w \in W_{n-1,k-1}$, prepend 0 to w. Hence $u_{n,k} = u_{n-1,k} + u_{n-1,k-1}$, as claimed.

Now suppose $k = n-1,\ n \geq 4$. Partition $W_{n,n-1}$ into 5 subsets:

$$\begin{aligned} B_1 &:= \{w \in W_{n,n-1} : w \text{ has no big entry}\}, \\ B_2 &:= \{w \in W_{n,n-1} : w \text{ has a big entry and first } n-2 \text{ entries are all 0's}\}, \\ B_3 &:= \{w \in W_{n,n-1} : w \text{ has a big entry, first } n-2 \text{ entries} \\ &\qquad\qquad \text{not all 0, } w_{n-1} = n-2,\ w_{n-2} > w_n\}, \\ B_4 &:= \{w \in W_{n,n-1} : w \text{ has a big entry, first } n-2 \text{ entries} \\ &\qquad\qquad \text{not all 0, } w_{n-1} = n-2,\ w_{n-2} \leq w_n\}, \\ B_5 &:= \{w \in W_{n,n-1} : w \text{ has a big entry, first } n-2 \text{ entries} \\ &\qquad\qquad \text{not all 0, } w_{n-1} \neq n-2\}. \end{aligned}$$

Note that in the subsets B_3 and B_4, the first condition is implied by the third condition $w_{n-1} = n-2$.

The map "delete first entry" is a bijection from B_1 to $W_{n-1,n-2}$. For example, $0001342 \to 001342$.

An element w of B_2 must have $w_{n-1} = n-2$ since w has a big entry, and $w_n \in [0, n-3]$ since w_{n-1} is a descent top. Hence $|B_2| = n-2$, a summand in the recurrence.

The map "switch last 2 entries" is a bijection from B_3 to $\{w \in W_{n,n-2} : w_n = n-2\}$. For example, $0002453 \to 0002435$. Also, the map "change w_{n-1} from $n-2$ to 0" is a bijection from B_4 to $\{w \in W_{n,n-2} : w_n \leq n-3,\ w_{n-1} = 0\}$. For example, $0002354 \to 0002304$.

Hence, it remains to show that B_5 renamed as U_n (to take account of sequence length) and

$$V_n := \{w \in W_{n,n-2} : w_n \leq n-3,\ w_{n-1} \neq 0\}$$

are equinumerous.

4.3.2. The Case of U_n and V_n

We refine U_n and V_n according to the size of the last descent: $U_{n,j} := \{w \in W_{n,n-1} : w \text{ has a big entry, first } n-2 \text{ entries not all 0, } w_{n-1} \neq n-2,\ w_{n-1} - w_n = j\}$ for $1 \leq j \leq n-5$. Similarly,

$$V_{n,j} := \{w \in W_{n,n-2} : w_n \leq n-3,\ w_{n-1} \neq 0,\ w_{n-2} - w_{n-1} = j\}$$

for $1 \leq j \leq n-5$.

We first show that $|U_{n+1,j+1}| = 2\,|U_{n,j}|$ and $|V_{n+1,j+1}| = 2\,|V_{n,j}|$, both for $1 \leq j \leq n-6$. Given $w \in U_{n,j}$, form two new sequences as follows: increment w_{n-1} by 1 and prepend 0 as in $0001042 \to 00001052$, or insert $w_{n-1}+1$ in position n as in $0001042 \to 00010452$. This is a 1-to-2 map from $U_{n,j}$ onto $U_{n+1,j+1}$. Similarly, given $w \in V_{n,j}$, form two new sequences: increment each of w_{n-2} and w_n by 1 and prepend 0 as in $0001324 \to 00001425$, or increment w_n by 1 and insert $w_{n-2}+1$ in position $n-1$ as in $0001324 \to 00013425$. This is a 1-to-2 map from $V_{n,j}$ onto $V_{n+1,j+1}$.

These observations reduce our problem to showing that $U_{n,1}$ and $V_{n,1}$ are equinumerous. Set $f(n,a) = \min(n-3-a, \lfloor (a-1)/2 \rfloor)$.

Proposition 2. *We have*
(i) For all $n \geq 1$,

$$|U_{n,1}| = \sum_{k=1}^{\lfloor n/3 \rfloor -1} \binom{n-k-4}{2k} 2^{n-3k-4} + \binom{n-k-4}{2k-1} 2^{n-3k-3}.$$

(ii) For all $n \geq 1$,

$$\begin{aligned} |V_{n,1}| &= \sum_{a=2}^{n-4} 2^{a-2}(n-a-3) \\ &\quad + \sum_{a=2}^{n-4} \sum_{k=1}^{f(n,a)} \left(\binom{a-2}{2k} 2^{a-2k-2} + \binom{a-2}{2k-1} 2^{a-2k-1} \right) (n-k-a-2) \\ &= 2^{n-4} - n + 3 \\ &\quad + \sum_{a=2}^{n-4} \sum_{k=1}^{f(n,a)} \left(\binom{a-2}{2k} 2^{a-2k-2} + \binom{a-2}{2k-1} 2^{a-2k-1} \right) (n-k-a-2). \end{aligned}$$

Proof. (i) Let $w \in U_{n,1}$. So the last descent top of w is in position $n-1$ and w has a big entry. Let j denote the position of the last big entry in w. If $j \leq n-2$, then by Lemma 1, w_n would be a big entry, contradicting the fact that w_{n-1} is a descent top. Hence $j = n-1$ and w_{n-1} is a big entry. Since $w_{n-1} \neq n-2$ and and w_{n-1} is a big entry, there must be at least one descent in $(w_i)_{i=1}^{n-2}$ for otherwise w_{n-1} would be $n-2$.

Suppose there are k descents in $(w_i)_{i=1}^{n-2}$. Then there are $(n-3)-k$ weak ascents in $(w_i)_{i=1}^{n-2}$ and since w_{n-1} is big, $w_{n-1} = n-k-2$. Hence, $w_n = w_{n-1} - 1 = n-k-3$ and all other entries of w are at most $n-k-4$.

It is now clear that w is determined by a $(2k)$-element subset X of $[0, n-k-4]$ to form the descent pairs in $(w_i)_{i=1}^{n-2}$ and an arbitrary subset Y of $[1, n-k-4] \setminus X$ to give the remaining nonzero entries in $(w_i)_{i=1}^{n-2}$. For example, when $n = 11$ and $k = 2$, if $X = \{0, 1, 2, 4\}$ and $Y = \emptyset$, then $w_{10} = n-k-2 = 7$ and $w = 00000104276$, while if $Y = \{3\}$, then $w = 00001034276$ and if $Y = \{3, 5\}$, then $w = 00010342576$.

If $0 \notin X$, there are $\binom{n-k-4}{2k}$ choices to select X and Y, is a subset of a set of $(n-k-4) - 2k = n-3k-4$ elements—2^{n-3k-4} choices. On the other hand, If $0 \in X$, there are $\binom{n-k-4}{2k-1}$ choices to select X, and Y is a subset of a set of $(n-k-4) - (2k-1) = n-3k-3$ elements—2^{n-3k-3} choices. Part (i) follows.

(ii) Let $w \in V_{n,1}$. Since w_{n-2} is a descent top, we have $w_{n-2} < w_n$, and by hypothesis, $w_n \leq n-3$. Since $w_{n-1} \neq 0$ and $w_{n-2} = w_{n-1} + 1$, it follows that $a := w_{n-2} \in [2, n-4]$. To meet the weak ascent condition, there are some restrictions on k and an additional condition on w_n, presented in the next paragraph. Then w is determined by a $(2k)$-element subset X of $[0,a-2]$ to form the descent pairs in $(w_i)_{i=1}^{n-3}$, an arbitrary subset Y of $[1,a-2]\setminus X$ to give the remaining nonzero entries in $(w_i)_{i=1}^{n-3}$, and a single element of $[a+1,n-3]$ to serve as w_n. For example, with $n=11$, $a=6$, $k=1$, if $X=2,3$ and $Y=\emptyset$ and $w_n=8$ (which are all permissible), then $w=0^6\,32658$ while if $Y=1,4$, then $w=0^4\,1324658$.

With k descents in $(w_i)_{i=1}^{n-3}$, there are $(n-4)-k$ weak ascents in $(w_i)_{i=1}^{n-3}$ and so $a=w_{n-2}\le n-k-3$, whence $k\le n-a-3$. In the interval $[0,a-2]$ there is room for at most $(a-1)/2$ disjoint pairs and so $k\le(a-1)/2$. Also, there are $n-k-3$ weak ascents in $(w_i)_{i=1}^{n-1}$ and we have $w_n\le n-k-2$ in addition to $w_n\le n-3$. When $k=0$, there are 2^{a-2} choices for Y, and $w_n\in[a+1,n-3]$, so $n-a-3$ choices for w_n. When $1\le k\le f(n,a):=\min(n-3-a,\lfloor(a-1)/2\rfloor)$, (1) when $0\notin X$, there are $\binom{a-2}{2k}$ choices for X and 2^{a-2k-2} choices for Y, and (2) when $0\in X$, there are $\binom{a-2}{2k-1}$ choices for X and $2^{a-2-(2k-1)}=2^{a-2k-1}$ choices for Y. In both cases (1) and (2), $a+1\le w_n\le n-k-2$ and so there are $n-k-a-2$ choices for w_n. Part (ii) follows. □

Lemma 2. *Let $u_n=|U_{n,1}|$ and $v_n=|V_{n,1}|$. Then, u_n and v_n both satisfy the recurrence relation $f_n=5f_{n-1}-7f_{n-2}+2f_{n-3}$ with $f_6=1$ and $f_j=0$ for all $j\le 5$.*

Proof. Let $u_n=|U_{n,1}|$ and $v_n=|V_{n,1}|$. By Proposition 2, we have

$$u_n=\sum_{k=1}^{\lfloor n/3\rfloor-1}\left(\binom{n-k-4}{2k}+2\binom{n-k-4}{2k-1}\right)2^{n-3k-4},$$

$$v_n=2^{n-4}-n+3+\sum_{a=2}^{n-4}\sum_{k=1}^{f(n,a)}\left(\binom{a-2}{2k}+2\binom{a-2}{2k-1}\right)(n-k-a-2)2^{a-2k-2}.$$

Let us show that $u_{3n+\ell}=5u_{3n+\ell-1}-7u_{3n+\ell-2}+2u_{3n+\ell-3}$ with $\ell=0,1,2$. By definitions, we have

$$\begin{aligned}
&-u_{3n}+5u_{3n-1}-7u_{3n-2}+2u_{3n-3}\\
&\quad=-1+\sum_{k=1}^{n-2}\left(-\binom{3n-k-4}{2k}-2\binom{3n-k-4}{2k-1}\right)2^{3n-3k-4}\\
&\qquad+\sum_{k=1}^{n-2}\left(5\binom{3n-k-5}{2k}+10\binom{3n-k-5}{2k-1}\right)2^{3n-1-3k-4}\\
&\qquad+\sum_{k=1}^{n-2}\left(-7\binom{3n-k-6}{2k}-14\binom{3n-k-6}{2k-1}\right)2^{3n-3k-6}\\
&\qquad+\sum_{k=1}^{n-2}\left(2\binom{3n-k-7}{2k}+4\binom{3n-k-7}{2k-1}\right)2^{3n-3k-7}
\end{aligned}$$

$$= -1 + \sum_{k=1}^{n-2} \frac{a_{n,k} 2^{3n-3k-6}(3n-k-7)!}{(3n-3k-3)!(2k-1)!},$$

where $a_{n,k} = 5k^3 - 15k^2n - 81kn^2 + 27n^3 + 36k^2 + 360kn - 108n^2 - 413k + 93n + 36$. By mathematical programming (such as Maple), we have that

$$\sum_{k=1}^{n-2} \frac{a_{n,k} 2^{3n-3k-6}(3n-k-7)!}{(3n-3k-3)!(2k-1)!} = 1.$$

Hence, the sequence u_n satisfies $u_{3n} = 5u_{3n-1} - 7u_{3n-2} + 2u_{3n-3}$. By using same argument, we obtain that u_n satisfies $u_{3n+\ell} = 5u_{3n+\ell-1} - 7u_{3n+\ell-2} + 2u_{3n+\ell-3}$ with $\ell = 1, 2$, which completes the proof for the sequence u_n.

By considering the same arguments, we can show that $v_{2n+\ell} = 5v_{2n+\ell-1} - 7v_{2n+\ell-2} + 2v_{2n+\ell-3}$ with $\ell = 0, 1$. □

Proposition 3. *Both sums in Proposition 2 evaluate to* $F_{2n-7} - 2^{n-4}$.

Proof. Let $u_n = |U_{n,1}|$ and $v_n = |V_{n,1}|$. By Lemma 2, we have that u_n and v_n both satisfy the recurrence relation $f_n = 5f_{n-1} - 7f_{n-2} + 2f_{n-3}$ with $f_6 = 1$ and $f_j = 0$ for all $j \le 5$. Hence, $u_n = v_n$.

Define $F(x) = \sum_{n\ge 6} f_n x^n$. Then the recurrence leads to

$$\begin{aligned} F(x) &= \frac{x^6}{(1-2x)(1-3x+x^2)} \\ &= -\frac{x^3}{2} - \frac{7x^2}{4} - \frac{39x}{8} - \frac{207}{16} - \frac{1}{16(1-2x)} + \frac{13-34x}{1-3x+x^2}, \end{aligned}$$

which, using $\frac{1}{1-3x+x^2} = \sum_{n\ge 0} F_{2n+2}x^n$ (F_n denotes the nth Fibonacci number) gives

$$f_n = 13F_{2n+2} - 34F_{2n} - 2^{n-4} = F_{2n-7} - 2^{n-4},$$

for all $n \ge 6$. □

4.4. Classes 61, 93, 99, 162, 205, 206, and 219

A *left to right maximum*, *LRmax* for short, in a sequence of nonnegative integers $a = a_1a_2\cdots a_n$ is an entry a_i such that $a_i > a_j$ for all $j < i$. Thus for $a = 011033101$, the LRmax entries are a_1, a_2, a_5 with values $0, 1, 3$, respectively. Any sequence of nonnegative integers can be decomposed uniquely as $m_1\pi^{(1)}\cdots m_k\pi^{(k)}$ where $m_1, \ldots, m_k$ are the LRmax entries in a, $m_1 < m_2 < \cdots < m_k$, and $m_i \ge \pi^{(i)}$ (entrywise). We call this the LRmax decomposition of a. If $a \in WA_n$, then also $m_1 = 0$ and so $\pi^{(1)}$ is all 0's, and $m_k \le n-1$.

We begin with Class 99.

Theorem 5 (Class 99)**.** *We have* $\{011, 100, 102, 201\} \overset{w}{\sim} \{011, 100, 120, 201\}$*.*

Proof. Suppose $a \in WA_n$ and $m_1\pi^{(1)}m_2\pi^{(2)} \cdots m_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in WA_n(011, 100, 102, 201)$ if and only if

- the nonzero entries in a are distinct;
- $\pi^{(i)} = \emptyset$, for all $i = 2, 3, \ldots, k-1$;
- $\pi^{(k)}$ is decreasing.

Also, $a \in WA_n(011, 100, 120, 201)$ if and only if

- $m_i > \pi^{(i)} > m_{i-1}$ for $i = 3, 4, \ldots, k$, and $m_2 > \pi^{(2)} \geq m_1 (= 0)$;
- $\pi^{(i)}$ is decreasing for $i = 2, 3, \ldots, k$.

To prove our theorem, we exhibit a bijection f from $a \in WA_n(011, 100, 102, 201)$ to $b \in WA_n(011, 100, 120, 201)$. For a, we can express $\pi^{(k)}$ as $\beta^{(k)}\beta^{(k-1)} \cdots \beta^{(2)}$, where $m_i > \beta^{(i)} > m_{i-1}$ for $i = 3, 4, \ldots, k$ and $m_2 > \beta^{(2)} \geq m_1 = 0$ and each $\beta^{(i)}$ is decreasing. Then define

$$\begin{aligned} b = f(a) &= f\left(m_1\pi^{(1)}m_2m_3m_4 \cdots m_k\beta^{(k)}\beta^{(k-1)} \cdots \beta^{(2)}\right) \\ &= m_1\pi^{(1)}m_2\beta^{(2)} \cdots m_k\beta^{(k)}. \end{aligned}$$

We leave the reader to verify that f is a bijection. □

Theorem 6 (Class 61)**.** *We have* $\{010, 011, 102, 201\} \overset{w}{\sim} \{010, 011, 120, 201\}$*.*

Proof. This class is related to Class 99. It is easy to check that $WA_n(010, 011, 102, 201) = \{a \in WA_n(011, 100, 102, 201) : 0 \text{ is not a descent bottom in } a\}$ and $WA_n(010, 011, 120, 201) = \{a \in WA_n(011, 100, 120, 201) : 0 \text{ is not a descent bottom in } a\}$. The bijection f of Theorem 5 for Class 99 preserves the "0 is not a descent bottom" property, and the theorem follows. □

It follows from the observations at the start of the preceding proof that Classes 99 and 61 are related by

$$|WA_n(011, 100, 102, 201)| = |WA_n(Q)| + |WA_{n-1}(Q)| - 1$$

for $n \geq 2$, where Q is the first quadruple in Class 61.

Theorem 7 (Class 93)**.** *We have* $\{011, 102, 120, 201\} \overset{w}{\sim} \{011, 102, 120, 210\}$*. Moreover, the generating function for the number of weak ascent sequences of length n that avoid* $\{011, 102, 120, 201\}$ *is given by* $\frac{x(2x^2-2x+1)}{(1-2x)(1-x)^2}$*.*

Proof. Let $A = \{011, 102, 120, 201\}$ and $B = \{011, 102, 120, 210\}$. We define $f : WA_n(A) \to WA_n(B)$ by mapping $\pi \in WA_n(A)$ to $\pi' \in WA_n(B)$, where π' is obtained from π by reversing the letters to the right of the first occurrence of the max letter of π. For example, if $\pi = 000012543$ then $\pi' = 000012534$. Clearly, f is a bijection, which shows that $A \stackrel{w}{\sim} B$.

Now, we find the generating function $F(x) := \sum_{n\geq 1} |WA_n(A)|x^n$. The sequences in $\cup_{n\geq 1} WA_n(A)$ can be characterized and counted directly by partitioning them into 3 subsets as follows:

- **No descents**: Clearly, the contribution to the generating function $F(x)$ is $\frac{x}{1-2x}$.
- **At least one descent, exactly one ascent**: These avoiders are a string of 0s followed by a decreasing sequence of 1 or more positive integers followed by a (possibly empty) string of 0s. It is easy to calculate that the contribution to the generating function $F(x)$ is $\frac{x^3(1+x)}{(1-x)^2(1-x-x^2)}$.
- **At least one descent, at least 2 ascents**: These avoiders are weak ascent sequences that consist of a string of 0s of length $i \geq 2$, followed by an increasing sequence of $j \geq 1$ positive integers ending at some integer a, followed by a decreasing sequence of $r \geq 2$ positive integers all greater than a. Hence, the contribution for the generating function $F(x)$ is given by the sum

$$\begin{aligned}
&\sum_{i\geq 2}\sum_{j\geq 1}\sum_{a=1}^{i+j-2} x^i \binom{a-1}{j-1} x^{j-1} \cdot x \cdot \sum_{r=2}^{i+j-a} \binom{i+j-a}{r} x^r \\
&= \sum_{i\geq 2}\sum_{j\geq 1}\sum_{a=1}^{i+j-2} \binom{a-1}{j-1} x^{i+j} \left((1+x)^{i+j-a} - (i+j-a)x - 1\right) \\
&= \sum_{j\geq 1}\sum_{a=1}^{j}\sum_{i\geq 2} \binom{a-1}{j-1} x^{i+j} \left((1+x)^{i+j-a} - (i+j-a)x - 1\right) \\
&\quad + \sum_{j\geq 1}\sum_{a\geq j+1}\sum_{i\geq a-j+2} \binom{a-1}{j-1} x^{i+j} \left((1+x)^{i+j-a} - (i+j-a)x - 1\right) \\
&= \frac{x^5}{(1-x)^3(1-x-x^2)} + \frac{x^6}{(1-x)^3(1-2x)(1-x-x^2)} \\
&= \frac{x^5}{(1-x)^2(1-x-x^2)(1-2x)}.
\end{aligned}$$

By adding all the contributions, we obtain that

$$F(x) = \frac{x}{1-2x} + \frac{x^3(1+x)}{(1-x)^2(1-x-x^2)} + \frac{x^5}{(1-x)^2(1-x-x^2)(1-2x)}$$

$$= \frac{x(2x^2 - 2x + 1)}{(1-2x)(1-x)^2},$$

as claimed. □

Theorem 8 (Class 162)**.** *We have* $\{010, 100, 101, 120\} \overset{w}{\sim} \{010, 100, 110, 120\}$.

Proof. Suppose $a \in WA_n$ and $m_1\pi^{(1)}m_2\pi^{(2)} \cdots m_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in WA_n(010, 100, 101, 120)$ if and only if

- $\pi^{(i)} > m_{i-1}$, for $i = 2, 3, \ldots, k$;
- $\pi^{(i)} = m_i \cdots m_i\beta^{(i)}$ such that $m_i > \beta^{(i)} > m_{i-1}$, for $i = 2, 3, \ldots, k$;
- $\beta^{(i)}$ avoids $\{00, 120\}$ for $i = 2, 3, \ldots, k$.

Also, $a \in WA_n(010, 100, 110, 120)$ if and only if

- $\pi^{(i)} > m_{i-1}$, for all $i = 2, 3, \ldots, k$;
- $\pi^{(i)} = \beta^{(i)}m_i \cdots m_i$ such that $m_i > \beta^{(i)} > m_{i-1}$, for all $i = 2, 3, \ldots, k$;
- $\beta^{(i)}$ avoids $\{00, 120\}$ for $i = 2, 3, \ldots, k$.

These two decompositions permit an obvious bijection between $WA_n(010, 100, 101, 120)$ and $WA_n(010, 100, 110, 120)$. □

Theorem 9 (Class 205)**.** *We have* $\{100, 102, 120, 201\} \overset{w}{\sim} \{102, 110, 120, 201\}$.

Proof. Suppose $a \in WA_n$ and $m_1\pi^{(1)}m_2\pi^{(2)} \cdots m_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in WA_n(100, 102, 120, 201)$ if and only if

- $\pi^{(i)} = m_i^{s_i}$ (superscripts denote repetition) with $s_i \geq 0$ for $i = 1, 2, \ldots, k-1$;
- $\pi^{(k)}$ either (i) $= m_k^{s_k}$ with $s_k \geq 0$ or (ii) $= m_k^{s_k}b_1 \ldots b_r m_k^{s'_k}$ with $m_k > b_1 > \cdots > b_r \geq m_{k-1}$, $r \geq 1$, $s_k \geq 0$, $s'_k \geq 0$ and $s'_k = 0$ unless $r = 1$.

Also, $a \in WA_n(102, 110, 120, 201)$ if and only if

- $\pi^{(i)} = m_i^{s_i}$ with $s_i \geq 0$ for $i = 1, 2, \ldots, k-1$;
- $\pi^{(k)}$ either (i) $= m_k^{s_k}$ with $s_k \geq 0$ or (ii) $= b_1 \ldots b_{r-1}b_r^{s_k}m_k^{s'_k}$ with $m_k > b_1 > \cdots > b_{r-1} > b_r \geq m_{k-1}$, $r \geq 1$, $s_k \geq 1$, $s'_k \geq 0$ and $s'_k = 0$ unless $r = 1$.

We can now define a bijection f from $a \in WA_n(100, 102, 120, 201)$ to $b \in WA_n(102, 110, 120, 201)$. Given a, if its $\pi^{(k)}$ falls in case (i) or case (ii) with $s_k = 0$, then a avoids 110 and $f(a) = a$, otherwise a contains 110, so delete the initial factor $m_k^{s_k}$ from $\pi^{(k)}$ and insert a factor $b_r^{s_k}$ next to b_r to change the 110s into 100s. This gives the $\pi^{(k)}$ for $b = f(a)$. For example, $000320 \to 000320$, $00011444321 \to 00011432111$, and $00011444244 \to 00011422244$. The inverse is clear. □

Theorem 10 (Class 206)**.** *We have*

$$\{100, 102, 120, 210\} \overset{w}{\sim} \{101, 102, 120, 201\} \overset{w}{\sim} \{101, 102, 120, 210\} \overset{w}{\sim} \{102, 110, 120, 210\}.$$

Proof. Let $B = \{100, 102, 120, 210\}, \{101, 102, 120, 201\}$, or $\{101, 102, 120, 210\}$. By our algorithm, the generating trees $\mathcal{T}'(B)$ has the following succession rules

$$\begin{aligned}
a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m};\\
b_{m,j} &\rightsquigarrow c_{m,j}, \dots, c_{m,1}, b_{m+1,j}, d_{m,j,j+1}, \dots, d_{m,j,m+1};\\
c_{m,j} &\rightsquigarrow c_{m+1,j-1}, \dots, c_{m+1,1}, c_{m+1,j};\\
d_{m,i,j} &\rightsquigarrow c_{m,j-i}, \dots, c_{m+1,1}, d_{m+1,i,j}, d_{m+1,j,j+1}, \dots, d_{m+1,j,m+2},
\end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j0$, $d_{m,i,j} = a_m ij$. Hence, the first 3 quadruples are *WA*-Wilf equivalent:

$$\{100, 102, 120, 210\} \overset{w}{\sim} \{101, 102, 120, 201\} \overset{w}{\sim} \{101, 102, 120, 210\}.$$

Now, we show that $\{101, 102, 120, 210\} \overset{w}{\sim} \{102, 110, 120, 210\}$. Suppose $a \in WA_n$ and $m_1\pi^{(1)}m_2\pi^{(2)} \cdots m_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in WA_n(101, 102, 120, 210)$ if and only if

- $\pi^{(i)} = m_i^{s_i}$ with $s_i \geq 0$ for $i = 1, 2, \dots, k-1$;
- $\pi^{(k)} = m_k^{s_k}\beta$ where $s_k \geq 0$ and β forms a nondecreasing sequence such that $m_k > \beta \geq m_{k-1}$.

Also, $b \in WA_n(102, 110, 120, 210)$ if and only if the following hold:

- $\pi^{(i)} = m_i^{s_i}$ with $s_i \geq 0$ for $i = 1, 2, \dots, k-1$;
- $\pi^{(k)} = \beta m_k^{s_k}$, where $s_k \geq 0$ and β is a nondecreasing sequence such that $m_k > \beta \geq m_{k-1}$.

Clearly, mapping the LRmax decomposition of $a \in WA_n(101, 102, 120, 210)$ to the LRmax decomposition of $b \in WA_n(102, 110, 120, 210)$ by suitably rearranging the entries of $\pi^{(k)}$, we obtain a bijection between $WA_n(101, 102, 120, 210)$ and $WA_n(102, 110, 120, 210)$. □

Theorem 11 (Class 219)**.** *We have* $\{100, 101, 102, 201\} \overset{w}{\sim} \{100, 110, 120, 201\}$.

Proof. Suppose $a \in WA_n$ and $m_1\pi^{(1)}m_2\pi^{(2)} \cdots m_k\pi^{(k)}$ is the LRmax decomposition of a. Then $a \in WA_n(100, 101, 102, 201)$ if and only if

- $\pi^{(i)} = m_i^{s_i}$ where $s_i \geq 0$ for $i = 1, 2, \dots, k-1$;

- $\pi^{(k)} = m_k^{s_k}\beta^{(k)}\beta^{(k-1)}\cdots\beta^{(2)}$ where $s_k \ge 0$ and $m_i > \beta^{(i)} \ge m_{i-1}$ and $\beta^{(i)}$ is a decreasing sequence, both for $i = 2, 3, \dots, k$;

Also, $b \in WA_n(100, 110, 120, 201)$ if and only if

- $\pi^{(i)} = \beta^{(i)} m_i^{s_i}$ where $s_i \ge 0$, $m_i > \beta^{(i)} \ge m_{i-1}$ and $\beta^{(i)}$ is a decreasing sequence, all for $i = 2, 3, \dots, k$.

These decompositions suggest the following bijection from $WA_n(100, 101, 102, 201)$ to $WA_n(100, 110, 120, 201)$:

$$m_1\pi^{(1)}m_2\pi^{(2)}\dots m_{k-1}\pi^{(k-1)}m_k m_k^{s_k}\beta^{(k)}\beta^{(k-1)}\dots\beta^{(2)} \to \\ m_1\pi^{(1)}m_2\beta^{(2)}\pi^{(2)}\dots m_{k-1}\beta^{(k-1)}\pi^{(k-1)}m_k\beta^{(k)}m_k^{s_k}.$$

For example, with vertical lines enclosing each LRmax,

$$|0|00|1|2|22|5|555|8|8876540 \to |0|00|1|0|2|22|5|4555|8|76588.$$

□

4.5. Class 211

Let $Q = \{101, 102, 110, 201\}$. Set $F(x) = \sum_{n\ge0} |WA_n(Q)|x^n$.

Theorem 12. *We have*

$$F(x) = \frac{(2 - 3x - x^2)\sqrt{1 - 4x} - x - x^2}{2(1 - x)(1 - 4x - x^2)}.$$

The rest of this section is devoted to the proof of Theorem 12. For $w = (w_i)_{i=1}^n \in A_n(Q)$, after a descent $w_i > w_{i+1}$, all later entries are at most w_{i+1} (to avoid 101, 102, 201) and if $w_{i+2} = w_{i+1}$, then $w_j = w_{i+1}$ for all $j \ge i + 1$ (to also avoid 110). If $w \in A_n(Q)$ has a descent, then $c := \max(w)$ occurs only once in w, to avoid 110, and c initiates the first descent.

Now partition $WA_n(Q)$ into 3 mutually exclusive classes:

- $X_n = \{w \in WA_n(Q) : w$ is weakly increasing (no descents)$\}$,
- $Y_n = \{w \in WA_n(Q) : w$ has at least one descent and w has no repeated entry before $c\}$,
- $Z_n = \{w \in WA_n(Q) : w$ has at least one descent and w has a repeated entry before $c\}$.

An avoider in Y_n has the form $0^j\,R\,c\,S\,s_0^k$ with $s_0 := \min(S)$ or $0^j\,R\,c\,S\,0^k$, in both cases with $R < c$ a strictly increasing sequence of positive integers and $S < c$ a strictly decreasing sequence of positive integers, and with $j \ge 1$ and $k \ge 0$. We partition Y_n into 3 subclasses according to the factors R and S :

- $Y_n^{(1)}$: $S = \emptyset$, so the avoider has the form $0^j\,R\,c\,0^k$ with $k \geq 1$,
- $Y_n^{(2)}$: $S \neq \emptyset$, $R \cap S = \emptyset$, so the avoider has the form either $0^j\,R\,c\,S\,s_0^k$ with $s_0 := \min(S)$ and $k \geq 0$, or $0^j\,R\,c\,S\,0^k$ with $k \geq 1$,
- $Y_n^{(3)}$: $S \neq \emptyset$, $R \cap S \neq \emptyset$.

For example,

$$\begin{aligned}
Y_3^{(1)} &= \{010\},\\
Y_4^{(1)} &= \{0010,\, 0020,\, 0100,\, 0120\},\\
Y_4^{(2)} &= \{0021\},\ Y_4^{(3)} = \{0121\},\\
Y_5^{(3)} &= \{00121,\, 00131,\, 00232,\, 01211,\, 01231,\, 01232\}.
\end{aligned}$$

For $w \in Z_n$, let a denote the largest repeated entry before c. Then w can be decomposed as (*) $P\,a^\ell\,R\,c\,S\,a^k$, $k \geq 1$, or (**) $P\,a^\ell\,R\,c\,S\,s_0^k$, $k \geq 0$, where in both cases, P is a weakly increasing sequence of nonnegative integers, $a \geq 1$, $\ell \geq 2$, R is strictly increasing (may be empty), S is strictly decreasing (may be empty in (*)), and $P < a < R < c > S > a$ and $s_0 := \min(S)$. Analogous to Y_n, we partition Z_n into 3 subclasses according to R and S :

- $Z_n^{(1)}$: case (*) with $S = \emptyset$, so the avoider form is
$$P\,a^\ell\,R\,c\,a^k,\ k \geq 1,$$
- $Z_n^{(2)}$: case (*) with $S \neq \emptyset$ and case (**) with $R \cap S = \emptyset$, so the avoider form is
$$P\,a^\ell\,R\,c\,S\,a^k,\ k \geq 1,\ R \cap S = \emptyset,\ S \neq \emptyset$$
or
$$P\,a^\ell\,R\,c\,S\,s_0^k,\ k \geq 0,\ R \cap S = \emptyset,\ S \neq \emptyset,$$
- $Z_n^{(3)}$: case (**) with $R \cap S \neq \emptyset$.

For example, $Z_5^{(1)} = \{01121,\ 01131\}$, $Z_5^{(2)} = \{01132\}$, $Z_5^{(3)} = \emptyset$, and $Z_6^{(3)} = \{011232,\ 011242,\ 011343\}$.

4.5.1. Generating Functions

Set $F_X(x) := \sum_{n \geq 0} |X_n|\,x^n$ and analogously for $F_Y(x)$ and $F_Z(x)$. Thus $F(x) = F_X(x) + F_Y(x) + F_Z(x)$.
`Case X`: We have $|X_n| = C_n$ and $F_X(x) = \frac{1-\sqrt{1-4x}}{2x}$.
`Case Y`: For $Y_n^{(1)}$, we have the form $w = 0^j\,R\,c\,0^k$ with $j \geq 1$, $k \geq 1$. Set $r = |R|$. Now $c \geq r+1$ since the entries in R are distinct. The latter fact also ensures that

w is in fact a weak ascent sequence provided only that c satisfies the weak ascent condition: $c \le r + j$. The generating function $F_{Y^{(1)}}(x) = \sum_{n\ge 3} |Y_n^{(1)}|\, x^n$ is thus given by

$$F_{Y^{(1)}}(x) = \sum_{j\ge 1}\sum_{r\ge 0}\sum_{c=r+1}^{r+j} \binom{c-1}{r} x^{j+r+1} \cdot \frac{x}{1-x} = \frac{x^3}{(1-x)^2(1-2x)}\,.$$

For $Y_n^{(2)}$, we have the form $w = 0^j\, R\, c\, S\, s_0^k$ with $k \ge 0$ or $w = 0^j\, R\, c\, S\, 0^k$ with $k \ge 1$. In both cases, $r = |R| \ge 0$ and $s = |S| \ge 1$ and $R \cap S = \emptyset$. Here $c \ge r + 2$ to allow room for $S \subseteq [1, c] \setminus R$. Again, w is a weak ascent sequence provided only that $c \le r + j$. The generating function $F_{Y^{(2)}}(x) = \sum_{n\ge 4} |Y_n^{(2)}|\, x^n$ is thus given by

$$\begin{aligned} F_{Y^{(2)}}(x) &= \sum_{j\ge 1}\sum_{r\ge 0}\sum_{c=r+2}^{r+j}\sum_{s=1}^{c-1-r} \binom{c-1}{r}\binom{c-1-r}{s} x^{j+r+1+s} \cdot \left(1 + \frac{2x}{1-x}\right) \\ &= \frac{x^4(1+x)}{(1-x)^2(1-2x)(1-2x-x^2)}\,. \end{aligned}$$

Here is a bijection ϕ_Y from Y_n to $Y_{n+1}^{(3)}$. Given $w \in Y_n$, set $c := \max(w)$ and $a := w_n$. If a occurs before c in w, let $w_j = a$ be the last occurrence of a before c, and then $\phi_Y(w) = 0\, w_1 \dots w_{j-1}\, (1 + w_i)_{i=j}^n$. For example, for $w = 00013540$, we have $j = 3$ and $a = 0$ and $\phi_Y(w) = 000124651$. On the other hand, if a does not occur before c in w, let j be maximal such that $w_j < a$. Then $\phi_Y(w) = (w_i)_{i=2}^j\, a\, (a + 1)\, (1 + w_i)_{i=j+1}^n$. For example, for $w = 0001422$, we have $j = 4$ and $a = 2$, and $\phi_Y(w) = 00123533$. Clearly, ϕ_Y preserves the weak ascent property and the two cases can be distinguished in the image $w' = \phi_Y(w)$ according to its last letter $a' = a + 1$: in the first case, $a' = 1$ or $a' > 1$ and $a' - 1$ does not appear in w', in the second case, $a' > 1$ and $a' - 1$ does appear in w'.

For $F_{Y^{(3)}}(x) := \sum_{n\ge 5} |Y_n^{(3)}|\, x^n$, the bijection ϕ_Y implies $F_{Y^{(3)}}(x) = xF_Y(x)$ and hence $F_{Y^{(3)}}(x) = x(F_{Y^{(1)}}(x) + F_{Y^{(2)}}(x))/(1 - x)$. It follows that

$$\begin{aligned} F_Y(x) &= F_{Y^{(1)}}(x) + F_{Y^{(2)}}(x) + F_{Y^{(3)}}(x) \\ &= F_{Y^{(1)}}(x) + F_{Y^{(2)}}(x) + \frac{x(F_{Y^{(1)}}(x) + F_{Y^{(2)}}(x))}{1-x} \\ &= \frac{F_{Y^{(1)}}(x) + F_{Y^{(2)}}(x)}{1-x}. \end{aligned}$$

`Case Z`: Define $F_{Z^{(1)}}(x)$, $F_{Z^{(2)}}(x)$, $F_{Z^{(3)}}(x)$ analogously to the Y case. Here it will be convenient to obtain a closed form expression for $F_{Z^{(12)}}(x) := F_{Z^{(1)}}(x) + F_{Z^{(2)}}(x)$ rather than for $F_{Z^{(1)}}(x)$, $F_{Z^{(2)}}(x)$ separately.

For $Z_n^{(1)}$, we have the form $w = P\, a^\ell\, R\, c\, a^k$ with $p := |P| \ge 1$, $\ell \ge 2$, $r = |R| \ge 0$, $k \ge 1$. Here P is a weakly increasing weak ascent sequence with maximum entry smaller than a, counted by $\binom{p+a-1}{p}\frac{p+2-a}{p+1}$. To ensure w is a weak ascent sequence, we need $c \le p+\ell+r$. So the generating function $F_{Z^{(1)}}(x)$ is given by the multisum

$$F_{Z^{(1)}}(x) = \sum_{p\ge 1}\sum_{a=1}^{p}\sum_{\ell\ge 2}\sum_{r\ge 0}\sum_{c=a+r+1}^{p+r+\ell}\binom{p+a-1}{p}\frac{p+2-a}{p+1}\binom{c-a-1}{r}\frac{x^{p+\ell+r+2}}{1-x}.$$

For $Z_n^{(2)}$, we have the form $w = P\,a^\ell\,R\,c\,S\,s_0^k,\ k \ge 0$ or $w = P\,a^\ell\,R\,c\,S\,a^k,\ k \ge 1$ with $p := |P| \ge 1$, $\ell \ge 2$, $r = |R| \ge 0$, $s := |S| >= 1$, $k \ge 1$. Here we get the generating function $F_{Z^{(2)}}(x)$ is given by the multisum

$$F_{Z^{(2)}}(x) = \sum_{p\ge 1}\sum_{a=1}^{p}\sum_{\ell\ge 2}\sum_{r\ge 0}\sum_{c=a+r+2}^{p+r+\ell}\sum_{s=1}^{c-a-r-1}\binom{p+a-1}{p}\frac{p+2-a}{p+1} \times \binom{c-a-1}{r}\binom{c-a-r-1}{s}x^{p+\ell+r+1+s}\left(1+\frac{2x}{1-x}\right).$$

Similar to the Y case, we have a bijection ϕ_Z from Z_n to $Z_{n+1}^{(3)}$. Given $w \in Z_n$, set $c := \max(w)$ and $a := w_n$. By definition of Z_n, there is at least one repeated entry in w before c, say b is the largest such. Then $b \le a$ to avoid 110. Let $w_j \ge b$ be the last occurrence before c of an entry at most a. Then $\phi_Z(w) = (w_i)_{i=1}^{j}\,(a+1)\,(1+w_i)_{i=j+1}^{n}$. For example, for $w = 00111563$ we have $a = 3$, $b = 1$ and $j = 5$, and $\phi_Z(w) = 001114674$. The inverse is clear.

For $F_{Z^{(3)}}(x)$, the bijection ϕ_Z implies $F_{Z^{(3)}}(x) = xF_Z(x)$ and hence

$$F_{Z^{(3)}}(x) = x(F_{Z^{(1)}}(x) + F_{Z^{(2)}}(x))/(1-x) = xF_{Z^{(12)}}(x)/(1-x).$$

It follows that

$$\begin{aligned} F_Z(x) &= F_{Z^{(1)}}(x) + F_{Z^{(2)}}(x) + F_{Z^{(3)}}(x) \\ &= F_{Z^{(12)}}(x) + \frac{xF_{Z^{(12)}}(x)}{1-x} \\ &= \frac{F_{Z^{(12)}}(x)}{1-x}. \end{aligned}$$

4.5.2. Finding the Generating Function $F_{Z^{(12)}}(x)$

From the preceding subsection, we have

$$F_{Z^{(1)}}(x) = \sum_{p\ge 1}\sum_{a=1}^{p}\sum_{\ell\ge 2}\sum_{r\ge 0}\sum_{c=a+r+1}^{p+r+\ell}\binom{p+a-1}{p}\frac{p+2-a}{p+1}\binom{c-a-1}{r}\frac{x^{p+\ell+r+2}}{1-x}$$

and

$$F_{Z^{(2)}}(x) = \sum_{p\ge 1}\sum_{a=1}^{p}\sum_{\ell\ge 2}\sum_{r\ge 0}\sum_{c=a+r+1}^{p+r+\ell}\binom{p+a-1}{p}\frac{(p+2-a)(1-(1+x)^{-c+r+a+1})}{(p+1)(1-x)(1+x)^{-c+r+a}}$$

$$\times \binom{c-a-1}{r} x^{p+\ell+r+1}.$$

Thus,

$$\begin{aligned}
&F_{Z^{(12)}}(x) = F_{Z^{(1)}}(x) + F_{Z^{(2)}}(x) \\
&= \sum_{p\geq 1}\sum_{a=1}^{p}\sum_{\ell\geq 2}\sum_{r\geq 0}\sum_{c=a+r+1}^{p+r+\ell} \frac{p+2-a}{p+1}\binom{p+a-1}{p}\binom{c-a-1}{r} \\
&\quad \times \frac{((1+x)^{c-r-a}-1)x^{p+\ell+r+1}}{1-x} \\
&= \sum_{p\geq 1}\sum_{a=1}^{p}\sum_{\ell\geq 2}\sum_{r\geq 0}\sum_{c=1}^{p+\ell-a} \frac{p+2-a}{p+1}\binom{p+a-1}{p}\binom{r+c-1}{r}\frac{((1+x)^{c}-1)x^{p+\ell+r+1}}{1-x} \\
&= \sum_{p\geq 1}\sum_{a=1}^{p}\sum_{\ell\geq 2}\sum_{c=1}^{p+\ell-a} \frac{p+2-a}{p+1}\binom{p+a-1}{p}\frac{((1+x)^{c}-1)x^{p+\ell+1}}{(1-x)^{c+1}}.
\end{aligned}$$

By simplifying the two innermost sums, we obtain

$$\begin{aligned}
F_{Z^{(12)}}(x) &= \sum_{p\geq 1}\sum_{a=1}^{p} \frac{(-\frac{1-2x}{1-x}+2(1-x)^{a-p-2})(p+2-a)x^{p+2}\binom{p+a-1}{a-1}}{2(p+1)(2x-1)} \\
&\quad - \sum_{p\geq 1}\sum_{a=1}^{p} \frac{(1+x)^3(1-x)^{a-p-2}(p+2-a)x^{p+2}\binom{p+a-1}{a-1}}{2(p+1)(x^2+2x-1)(1+x)^{p-a}}.
\end{aligned}$$

Using the identity

$$\sum_{p\geq 1}\sum_{a=1}^{p} \frac{p+2-a}{p+1}\binom{p+a-1}{a-1} x^{p+2}y^{p+2-a} = \frac{x^3y^2C^2(x)}{1-xyC(x)}$$

with $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, we obtain

$$\begin{aligned}
F_{Z^{(12)}}(x) &= \frac{(1-3x)\sqrt{1-4x}}{2x(1-4x-x^2)} \\
&\quad + \frac{4x^7+12x^6-12x^5-23x^4+54x^3-36x^2+10x-1}{2x(1-2x)(1-x)(1-2x-x^2)(1-4x-x^2)}.
\end{aligned}$$

Theorem 12 now follows by adding the expressions obtained for F_X, F_Y and F_Z in the preceding subsections.

4.6. Class 226

By our algorithm, the generating trees

$$\mathcal{T}'(100, 110, 201, 210) \text{ and } \mathcal{T}'(101, 110, 201, 210)$$

have a root a_1 and satisfy the following succession rules:

$$\begin{aligned}
a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \dots, b_{m,m}, \quad m \geq 1; \\
b_{1,1} &\rightsquigarrow 010, c_{1,1}, b_{2,2}; \\
010 &\rightsquigarrow c_{1,1}, e_1; \\
b_{m,j} &\rightsquigarrow (c_{m-1,j})^j, c_{m,j}, b_{m+1,j+1}, \dots, b_{m+1,m+1}, \quad 1 \leq j \leq m-1; \\
b_{m,m} &\rightsquigarrow (f_m)^m, c_{m,m}, b_{m+1,m+1}, \quad m \geq 2; \\
c_{m,j} &\rightsquigarrow c_{m+1,j}, d_{m,j,j+1}, \dots, d_{m,j,m+1}; \\
e_m &\rightsquigarrow f_{m+1}, c_{m+1,m+1}, d_{m,m,m+2}; \\
f_m &\rightsquigarrow c_{m,m}, e_m; \\
d_{m,i,j} &\rightsquigarrow (c_{m+1,j})^{j-i}, c_{m+2,j}, d_{m+1,i,j+1}, \dots, d_{m+1,i,m+3}, i+1 \leq j \leq m+1; \\
d_{m,i,m+2} &\rightsquigarrow (f_{m+2})^{m+2-i}, c_{m+2,m+2}, d_{m+1,i,m+3},
\end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m jj$, $d_{m,i,j} = a_m iij$, $e_m = a_m m0(m+1)$, and $f_m = a_m m0$. Hence,

$$\mathcal{T}'(100, 110, 201, 210) = \mathcal{T}'(101, 110, 201, 210),$$

which implies that $\{100, 110, 201, 210\} \overset{w}{\sim} \{101, 110, 201, 210\}$.

5. Further Results

In [7] it is stated that by using our algorithm, one can show that the number AW_k of A-Wilf-equivalence classes of k length-3 patterns, $k \geq 5$, is given by

$$\begin{aligned}
&AW_5 = 61, \quad AW_6 = 47, \quad AW_7 = 35, \quad AW_8 = 25, \quad AW_9 = 18, \\
&AW_{10} = 12, \quad AW_{11} = 7, \quad AW_{12} = 3, \quad AW_{13} = 1.
\end{aligned}$$

Also, by using our algorithm, one can show that the number WAW_k of WA-Wilf-equivalence classes of k length-3 patterns, $k \geq 5$, is given by

$$\begin{aligned}
&WAW_5 = 231, \quad WAW_6 = 171, \quad WAW_7 = 104, \quad WAW_8 = 59, \quad WAW_9 = 35, \\
&WAW_{10} = 21, \quad WAW_{11} = 10, \quad WAW_{12} = 4, \quad WAW_{13} = 1.
\end{aligned}$$

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Appendix A

Table 4: Ascent sequences avoiding a quadruples of length-3 patterns.

Beginning of Table 4		
Class	B quadruple	$\{\lvert A_n(B)\rvert\}_{n=1}^{10}$
1	{000,001,010,012},{000,010,011,012},{000,001,011,012}	1,2,1,0,0,0,0,0,0,0
2	{000,001,010,011},{001,010,011,012}	1,2,1,1,1,1,1,1,1,1
3	{000,011,012,100},{000,011,012,021},{000,011,012,101} {000,011,012,102},{000,011,012,110},{000,011,012,120} {000,011,012,201},{000,011,012,210},{000,001,012,110}	1,2,2,0,0,0,0,0,0,0
4	{000,010,012,021},{000,010,012,100},{000,010,012,101} {000,010,012,102},{000,010,012,110},{000,010,012,120} {000,010,012,201},{000,010,012,210},{000,001,012,100} {000,001,012,101},{000,001,012,021},{000,001,012,102} {000,001,012,120},{000,001,012,201},{000,001,012,210}	1,2,2,1,0,0,0,0,0,0
5	{001,011,012,100},{000,001,011,120}	1,2,2,1,1,1,1,1,1,1
6	{000,001,010,021},{000,001,010,100},{000,001,010,101} {000,001,010,102},{000,001,010,110},{000,001,010,120} {000,001,010,201},{000,001,010,210},{000,010,011,021} {000,010,011,100},{000,010,011,101},{000,010,011,102} {000,010,011,110},{000,010,011,120},{000,010,011,201} {000,010,011,210},{001,010,011,021},{001,010,011,100} {001,010,011,101},{001,010,011,102},{001,010,011,110} {001,010,011,120},{001,010,011,201},{001,010,011,210} {001,010,012,021},{001,010,012,100},{001,010,012,101} {001,010,012,102},{001,010,012,110},{001,010,012,120} {001,010,012,201},{001,010,012,210},{001,011,012,021} {001,011,012,101},{001,011,012,102},{001,011,012,110} {001,011,012,120},{001,011,012,201},{001,011,012,210} {010,011,012,021},{010,011,012,100},{010,011,012,101} {010,011,012,102},{010,011,012,110},{010,011,012,120} {010,011,012,201},{010,011,012,210},{000,001,011,102} {000,001,011,100},{000,001,011,021},{000,001,011,101} {000,001,011,110},{000,001,011,201},{000,001,011,210}	1,2,2,2,2,2,2,2,2,2
T7	000,012,101,110	1,2,3,1,0,0,0,0,0,0
8	{000,012,100,101},{000,012,100,110},{000,012,021,101} {000,012,101,102},{000,012,101,120},{000,012,101,201} {000,012,101,210},{000,012,021,110},{000,012,102,110} {000,012,110,120},{000,012,110,201},{000,012,110,210}	1,2,3,2,0,0,0,0,0,0
9	{000,011,102,120},{001,011,100,120},{001,012,100,110}	1,2,3,2,2,2,2,2,2,2
10	{000,012,021,100},{000,012,100,102},{000,012,100,120} {000,012,100,201},{000,012,100,210},{000,012,021,102} {000,012,021,120},{000,012,021,201},{000,012,021,210} {000,012,102,120},{000,012,102,201},{000,012,102,210} {000,012,120,201},{000,012,120,210},{000,012,201,210}	1,2,3,3,0,0,0,0,0,0
T11	{000,001,021,120}	1,2,3,3,2,2,2,2,2,2
12	{000,011,100,102},{000,011,100,120},{001,011,100,102} {001,011,102,120},{001,012,100,101},{001,012,101,110} {011,012,021,100},{011,012,100,101},{011,012,100,102} {011,012,100,110},{011,012,100,120},{011,012,100,201} {011,012,100,210},{000,011,021,102},{000,011,101,102} {000,011,102,110},{000,011,102,201},{000,011,102,210} {000,011,021,120},{000,011,101,120},{000,011,110,120} {000,011,120,201},{000,011,120,210},{001,011,021,100} {001,011,021,120},{001,011,100,101},{001,011,100,110} {001,011,100,201},{001,011,100,210},{001,011,101,120} {001,011,110,120},{001,011,120,201},{001,011,120,210} {001,012,021,100},{001,012,021,110},{001,012,100,102} {001,012,100,120},{001,012,100,201},{001,012,100,210} {001,012,102,110},{001,012,110,120},{001,012,110,201} {001,012,110,210},{000,001,110,120},{000,001,021,110}	1,2,3,3,3,3,3,3,3,3
13	{000,001,101,120},{000,001,100,120},{000,001,102,120} {000,001,021,102},{000,001,120,201},{000,001,120,210}	

Continuation of Table 4		
Class	B quadruple	$\{\|A_n(B)\|\}_{n=1}^{10}$
	{000,001,021,101},{000,001,021,100},{000,001,021,210} {000,001,021,201}	1,2,3,4,4,4,4,4,4,4
14	{000,011,021,100},{000,011,100,101},{000,011,100,110 {000,011,100,201},{000,011,100,210},{001,010,021,100} {001,010,021,101},{001,010,021,102},{001,010,021,110} {001,010,021,120},{001,010,021,201},{001,010,021,210} {001,010,100,101},{001,010,100,102},{001,010,100,110} {001,010,100,120},{001,010,100,201},{001,010,100,210} {001,010,101,102},{001,010,101,110},{001,010,101,120} {001,010,101,201},{001,010,101,210},{001,010,102,110} {001,010,102,120},{001,010,102,201},{001,010,102,210} {001,010,110,120},{001,010,110,201},{001,010,110,210} {001,010,120,201},{001,010,120,210},{001,010,201,210} {001,011,021,102},{001,011,101,102},{001,011,102,110} {001,011,102,201},{001,011,102,210},{001,012,021,101} {001,012,101,102},{001,012,101,120},{001,012,101,201} {001,012,101,210},{010,011,021,100},{010,011,021,101} {010,011,021,102},{010,011,021,110},{010,011,021,120} {010,011,021,201},{010,011,021,210},{010,011,100,101} {010,011,100,102},{010,011,100,110},{010,011,100,120} {010,011,100,201},{010,011,100,210},{010,011,101,102} {010,011,101,110},{010,011,101,120},{010,011,101,201} {010,011,101,210},{010,011,102,110},{010,011,102,120} {010,011,102,201},{010,011,102,210},{010,011,110,120} {010,011,110,201},{010,011,110,210},{010,011,120,201} {010,011,120,210},{010,011,201,210},{010,012,021,100} {010,012,021,101},{010,012,021,102},{010,012,021,110} {010,012,021,120},{010,012,021,201},{010,012,021,210} {010,012,100,101},{010,012,100,102},{010,012,100,110} {010,012,100,120},{010,012,100,201},{010,012,100,210} {010,012,101,102},{010,012,101,110},{010,012,101,120} {010,012,101,201},{010,012,101,210},{010,012,102,110} {010,012,102,120},{010,012,102,201},{010,012,102,210} {010,012,110,120},{010,012,110,201},{010,012,110,210} {010,012,120,201},{010,012,120,210},{010,012,201,210} {011,012,021,101},{011,012,021,102},{011,012,021,110} {011,012,021,120},{011,012,021,201},{011,012,021,210} {011,012,101,102},{011,012,101,110},{011,012,101,120} {011,012,101,201},{011,012,101,210},{011,012,102,110} {011,012,102,120},{011,012,102,201},{011,012,102,210} {011,012,110,120},{011,012,110,201},{011,012,110,210} {011,012,120,201},{011,012,120,210},{011,012,201,210} {000,011,021,101},{000,011,021,110},{000,011,021,201} {000,011,021,210},{000,011,101,110},{000,011,101,201} {000,011,101,210},{000,011,110,201},{000,011,110,210} {000,011,201,210},{001,011,021,101},{001,011,021,110} {001,011,021,201},{001,011,021,210},{001,011,101,110} {001,011,101,201},{001,011,101,210},{001,011,110,201} {001,011,110,210},{001,011,201,210},{001,012,021,102} {001,012,021,120},{001,012,021,201},{001,012,021,210} {001,012,102,120},{001,012,102,201},{001,012,102,210} {001,012,120,201},{001,012,120,210},{001,012,201,210} {000,001,102,110},{000,001,101,110},{000,001,100,110} {000,001,110,201},{000,001,110,210}	1,2,3,4,5,6,7,8,9,10
15	{000,001,102,210},{000,001,101,210},{000,001,100,210} {000,001,201,210}	1,2,3,5,7,9,11,13,15,17
16	{000,010,021,100},{000,010,021,101},{000,010,021,102} {000,010,021,110},{000,010,021,120},{000,010,021,201} {000,010,021,210},{000,010,100,101},{000,010,100,102} {000,010,100,110},{000,010,100,120},{000,010,100,201} {000,010,100,210},{000,010,101,102},{000,010,101,110} {000,010,101,120},{000,010,101,201},{000,010,101,210} {000,010,102,110},{000,010,102,120},{000,010,102,201} {000,010,102,210},{000,010,110,120},{000,010,110,201}	

Continuation of Table 4

Class	B quadruple	$\{\lvert A_n(B)\rvert\}_{n=1}^{10}$
	{000,010,110,210},{000,010,120,201},{000,010,120,210} {000,010,201,210},{000,001,100,102},{000,001,101,102} {000,001,100,101},{000,001,102,201},{000,001,101,201} {000,001,100,201}	1,2,3,5,8,13,21,34,55,89
T17	{000,100,201,210}	1,2,4,10,25,66,176,479,1318,3670
18	{011,100,102,120},{012,100,101,110},{001,100,110,120} {001,021,100,120},{001,021,110,120},{001,021,100,110}	1,2,4,5,6,7,8,9,10,11
19	{011,021,100,102},{011,021,100,120},{011,021,102,120} {011,100,101,102},{011,100,101,120},{011,100,102,110} {011,100,102,201},{011,100,102,210},{011,100,110,120} {011,100,120,201},{011,100,120,210},{011,101,102,120} {011,102,110,120},{011,102,120,201},{011,102,120,210} {012,021,100,101},{012,021,100,110},{012,021,101,110} {012,100,101,102},{012,100,101,120},{012,100,101,201} {012,100,101,210},{012,100,102,110},{012,100,110,120} {012,100,110,201},{012,100,110,210},{012,101,102,110} {012,101,110,120},{012,101,110,201},{012,101,110,210} {001,100,102,120},{001,102,110,120},{001,100,101,120} {001,101,110,120},{001,100,102,110},{001,100,101,110} {001,021,101,120},{001,021,102,120},{001,021,100,102} {001,100,120,201},{001,100,120,210},{001,021,102,110} {001,021,100,101},{001,110,120,201},{001,110,120,210} {001,021,101,110},{001,100,110,201},{001,100,110,210} {001,021,120,201},{001,021,120,210},{001,021,100,210} {001,021,100,201},{001,021,110,210},{001,021,110,201}	1,2,4,6,8,10,12,14,16,18
20	{001,021,201,210},{001,101,102,120},{011,021,100,101} {011,021,100,110},{011,021,100,201},{011,021,100,210} {011,021,101,102},{011,021,102,110},{011,021,102,201} {011,021,102,210},{011,100,101,110},{011,100,101,201} {011,100,101,210},{011,100,110,201},{011,100,110,210} {011,100,201,210},{011,101,102,110},{011,101,102,201} {011,101,102,210},{011,102,110,201},{011,102,110,210} {011,102,201,210},{012,021,100,102},{012,021,100,120} {012,021,100,201},{012,021,100,210},{012,021,101,102} {012,021,101,120},{012,021,101,201},{012,021,101,210} {012,100,102,120},{012,100,102,201},{012,100,102,210} {012,100,120,201},{012,100,120,210},{012,100,201,210} {012,101,102,120},{012,101,102,201},{012,101,102,210} {012,101,120,201},{012,101,120,210},{012,101,201,210} {011,021,101,120},{011,021,110,120},{011,021,120,201} {011,021,120,210},{011,101,110,120},{011,101,120,201} {011,101,120,210},{011,110,120,201},{011,110,120,210} {011,120,201,210},{012,021,102,110},{012,021,110,120} {012,021,110,201},{012,021,110,210},{012,102,110,120} {012,102,110,201},{012,102,110,210},{012,110,120,201} {012,110,120,210},{012,110,201,210},{001,101,102,110} {001,101,120,201},{001,101,120,210},{001,102,120,201} {001,102,120,210},{001,102,110,201},{001,100,102,210} {001,102,110,210},{001,100,101,210},{001,021,101,102} {001,101,110,201},{001,101,110,210},{001,021,102,210} {001,021,102,201},{001,021,101,201},{001,021,101,210} {001,100,201,210},{001,120,201,210},{001,110,201,210}	1,2,4,7,11,16,22,29,37,46
21	{000,101,102,120},{000,102,110,120},{000,021,101,120} {000,021,101,102}	1,2,4,7,11,18,29,47,76,123
22	{001,100,101,102},{001,100,102,201},{001,100,101,201} {000,101,102,110},{000,021,101,110}	1,2,4,7,12,20,33,54,88,143
T23	{000,021,110,120}	1,2,4,7,12,21,36,62,106,181
T24	{000,101,110,120}	1,2,4,7,13,24,44,81,149,274
T25	{000,021,102,120}	1,2,4,7,8,13,21,34,55,89
T26	{000,021,102,110}	1,2,4,7,9,14,22,35,56,90
27	{000,021,102,201},{000,021,102,210},{000,021,100,102}	1,2,4,8,11,18,29,47,76,123
28	{000,100,102,120},{000,102,120,201},{000,102,120,210	1,2,4,8,12,20,32,52,84,136
T29	{000,102,110,201}	1,2,4,8,13,22,36,59,96,156

Continuation of Table 4		
Class	B quadruple	$\{\lvert A_n(B)\rvert\}_{n=1}^{10}$
30	{000,021,120,201},{000,021,120,210},{000,021,100,120}	1,2,4,8,13,23,39,67,114,194
31	{000,100,102,110},{000,102,110,210}	1,2,4,8,14,24,40,66,108,176
32	{001,101,102,210},{001,102,201,210},{001,101,201,210}	1,2,4,8,15,26,42,64,93,130
33	{000,021,101,201},{000,021,101,210},{000,021,110,210} {000,021,110,201},{000,101,102,210},{000,021,100,101} {000,021,100,110}	1,2,4,8,15,28,51,92,164,290
34	{000,100,101,120},{000,101,120,201},{000,101,120,210}	1,2,4,8,15,29,56,108,208,401
35	{001,101,102,201},{010,021,100,101},{010,021,100,102} {010,021,100,110},{010,021,100,120},{010,021,100,201} {010,021,100,210},{010,021,101,102},{010,021,101,110} {010,021,101,120},{010,021,101,201},{010,021,101,210} {010,021,102,110},{010,021,102,120},{010,021,102,201} {010,021,102,210},{010,021,110,120},{010,021,110,201} {010,021,110,210},{010,021,120,201},{010,021,120,210} {010,021,201,210},{010,100,101,102},{010,100,101,110} {010,100,101,120},{010,100,101,201},{010,100,101,210} {010,100,102,110},{010,100,102,120},{010,100,102,201} {010,100,102,210},{010,100,110,120},{010,100,110,201} {010,100,110,210},{010,100,120,201},{010,100,120,210} {010,100,201,210},{010,101,102,110},{010,101,102,120} {010,101,102,201},{010,101,102,210},{010,101,110,120} {010,101,110,201},{010,101,110,210},{010,101,120,201} {010,101,120,210},{010,101,201,210},{010,102,110,120} {010,102,110,201},{010,102,110,210},{010,102,120,201} {010,102,120,210},{010,102,201,210},{010,110,120,201} {010,110,120,210},{010,110,201,210},{010,120,201,210} {011,021,101,110},{011,021,101,201},{011,021,101,210} {011,021,110,201},{011,021,110,210},{011,021,201,210} {011,101,110,201},{011,101,110,210},{011,101,201,210} {011,110,201,210},{012,021,102,120},{012,021,102,201} {012,021,102,210},{012,021,120,201},{012,021,120,210} {012,021,201,210},{012,102,120,201},{012,102,120,210} {012,102,201,210},{012,120,201,210},{000,100,101,102} {000,100,101,110},{000,101,102,201},{000,101,110,201} {000,101,110,210}	1,2,4,8,16,32,64,128,256,512
T36	{000,100,110,120}	1,2,4,8,17,37,83,191,448,1072
T37	{000,110,120,201}	1,2,4,8,17,37,84,194,458,1097
T38	{000,110,120,210}	1,2,4,8,17,37,84,194,460,1110
T39	{000,102,201,210}	1,2,4,9,16,30,54,97,172,303
T40	{000,100,102,210}	1,2,4,9,17,33,61,112,202,361
T41	{000,100,102,201}	1,2,4,9,17,35,69,139,277,555
42	{000,021,100,201},{000,021,100,210},{000,021,201,210}	1,2,4,9,18,37,73,143,275,523
T43	{000,100,120,201}	1,2,4,9,19,43,98,230,545,1313
T44	{000,100,120,210}	1,2,4,9,19,43,98,230,547,1326
T45	{000,120,201,210}	1,2,4,9,19,43,99,235,562,1370
46	{000,101,201,210},{000,100,101,210}	1,2,4,9,20,45,101,227,510,1146
T47	{000,100,101,201}	1,2,4,9,21,51,127,323,835,2188
T48	{000,100,110,201}	1,2,4,9,21,51,127,324,843,2230
T49	{000,110,201,210}	1,2,4,9,21,51,128,330,868,2318
T50	{000,100,110,210}	1,2,4,9,22,58,163,484,1507,4890
T51	{021,100,102,110}	1,2,5,11,21,39,73,139,269,527
52	{021,100,102,120},{021,101,102,120},{021,102,110,120} {021,101,102,110}	1,2,5,11,22,42,79,149,284,548
53	{100,101,102,120},{100,102,110,120},{101,102,110,120} {100,101,102,110},{021,100,101,120},{021,101,110,120} {021,100,101,102},{021,100,101,110}	1,2,5,11,23,47,95,191,383,767
T54	{021,100,110,120}	1,2,5,11,24,52,112,240,512,1088
T55	{100,101,110,120}	1,2,5,11,24,53,117,258,569,1255
56	{021,102,110,201},{021,102,110,210}	1,2,5,12,25,48,89,164,305,576
57	{021,100,102,201},{021,100,102,210}	1,2,5,12,26,53,105,206,404,795
T58	{100,102,110,201}	1,2,5,12,26,54,110,222,446,894
T59	{100,102,110,210}	1,2,5,12,27,57,117,237,477,957
60	{100,102,120,201},{100,102,120,210},{101,102,120,201}	

Continuation of Table 4		
Class	B quadruple	$\{\lvert A_n(B)\rvert\}_{n=1}^{10}$
	{101,102,120,210},{101,102,110,201},{102,110,120,201} {102,110,120,210},{101,102,110,210},{021,102,120,201} {021,102,120,210},{021,101,102,201},{021,101,120,201} {021,101,120,210},{021,101,102,210}	1,2,5,12,27,58,121,248,503,1014
61	{100,101,102,210},{021,100,120,201},{021,100,120,210} {021,101,110,201},{021,100,101,201},{021,100,101,210} {021,101,110,210},{021,100,110,201},{021,100,110,210} {021,110,120,201},{021,110,120,210}	1,2,5,12,28,64,144,320,704,1536
62	{100,101,120,201},{100,101,120,210},{101,110,120,201} {101,110,120,210},{100,101,110,201},{100,101,110,210}	1,2,5,12,28,65,151,351,816,1897
T63	{100,101,102,201}	1,2,5,12,29,70,169,408,985,2378
T64	{100,110,120,201}	1,2,5,12,30,78,209,574,1610,4596
T65	{100,110,120,210}	1,2,5,12,30,78,209,575,1620,4659
T66	{102,110,201,210}	1,2,5,13,31,69,147,305,623,1261
T67	{021,102,201,210}	1,2,5,13,32,74,163,347,722,1480
T68	{100,102,201,210}	1,2,5,13,32,75,170,377,824,1783
69	{101,102,201,210},{102,120,201,210}	1,2,5,13,33,81,193,449,1025,2305
T70	{101,120,201,210}	1,2,5,13,33,82,202,497,1224,3017
71	{021,100,201,210},{021,110,201,210}	1,2,5,13,34,88,224,560,1376,3328
72	{100,101,201,210},{101,110,201,210},{021,101,201,210} {021,120,201,210}	1,2,5,13,34,89,233,610,1597,4181
73	{100,120,201,210},{110,120,201,210}	1,2,5,13,35,98,283,837,2524,7733
T74	{100,110,201,210}	1,2,5,13,35,98,284,846,2576,7984
End of Table 4		

Appendix B

Table 5: Weak ascent sequences avoiding a quadruples of length-3 patterns.

Beginning of Table 5		
Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
1	{000,001,010,012},{000,001,011,012}	1,2,1,0,0,0,0,0,0,0
2	{000,001,010,011},{001,010,011,012}	1,2,1,1,1,1,1,1,1,1
T3	{000,001,012,110}	1,2,2,0,0,0,0,0,0,0
4	{000,010,011,012},{000,001,012,100},{000,001,012,101} {000,001,012,021},{000,001,012,102},{000,001,012,120} {000,001,012,201},{000,001,012,210}	1,2,2,1,0,0,0,0,0,0
5	{001,011,012,100},{000,001,011,120}	1,2,2,1,1,1,1,1,1,1
6	{000,001,010,021},{000,001,010,100},{000,001,010,101} {000,001,010,102},{000,001,010,110},{000,001,010,120} {000,001,010,201},{000,001,010,210},{001,010,011,021} {001,010,011,100},{001,010,011,101},{001,010,011,102} {001,010,011,110},{001,010,011,120},{001,010,011,201} {001,010,011,210},{001,010,012,021},{001,010,012,100} {001,010,012,101},{001,010,012,102},{001,010,012,110} {001,010,012,120},{001,010,012,201},{001,010,012,210} {001,011,012,021},{001,011,012,101},{001,011,012,102} {001,011,012,110},{001,011,012,120},{001,011,012,201} {001,011,012,210},{000,001,011,102},{000,001,011,100} {000,001,011,021},{000,001,011,101},{000,001,011,110} {000,001,011,201},{000,001,011,210}	1,2,2,2,2,2,2,2,2,2
T7	{000,011,012,021}	1,2,3,0,0,0,0,0,0,0
8	{000,011,012,100},{000,011,012,101},{000,011,012,102} {000,011,012,110},{000,011,012,120},{000,011,012,201} {000,011,012,210	}1,2,3,1,0,0,0,0,0,0
T9	{000,010,012,021}	1,2,3,2,0,0,0,0,0,0
10	{001,011,100,120},{001,012,100,110}	1,2,3,2,2,2,2,2,2,2
11	{000,010,012,100},{000,010,012,110}	1,2,3,3,1,0,0,0,0,0

Continuation of Table 5

Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
12	{000,010,012,101},{000,010,012,102},{000,010,012,120} {000,010,012,201},{000,010,012,210}	1,2,3,3,2,1,0,0,0,0
T13	{000,001,021,120}	1,2,3,3,2,2,2,2,2,2
14	{001,011,100,102},{001,011,102,120},{001,012,100,101} {001,012,101,110},{001,011,021,100},{001,011,021,120} {001,011,100,101},{001,011,100,110},{001,011,100,201} {001,011,100,210},{001,011,101,120},{001,011,110,120} {001,011,120,201},{001,011,120,210},{001,012,021,100} {001,012,021,110},{001,012,100,102},{001,012,100,120} {001,012,100,201},{001,012,100,210},{001,012,102,110} {001,012,110,120},{001,012,110,201},{001,012,110,210} {000,001,110,120},{000,001,021,110}	1,2,3,3,3,3,3,3,3,3
15	{000,001,101,120},{000,001,100,120},{000,001,102,120} {000,001,021,102},{000,001,120,201},{000,001,120,210} {000,001,021,101},{000,001,021,100},{000,001,021,210} {000,001,021,201}	1,2,3,4,4,4,4,4,4,4
16	{000,010,011,021},{010,011,012,021},{001,010,021,100} {001,010,021,101},{001,010,021,102},{001,010,021,110} {001,010,021,120},{001,010,021,201},{001,010,021,210} {001,010,100,101},{001,010,100,102},{001,010,100,110} {001,010,100,120},{001,010,100,201},{001,010,100,210} {001,010,101,102},{001,010,101,110},{001,010,101,120} {001,010,101,201},{001,010,101,210},{001,010,102,110} {001,010,102,120},{001,010,102,201},{001,010,102,210} {001,010,110,120},{001,010,110,201},{001,010,110,210} {001,010,120,201},{001,010,120,210},{001,010,201,210} {001,011,021,102},{001,011,101,102},{001,011,102,110} {001,011,102,201},{001,011,102,210},{001,012,021,101} {001,012,101,102},{001,012,101,120},{001,012,101,201} {001,012,101,210},{001,011,021,101},{001,011,021,110} {001,011,021,201},{001,011,021,210},{001,011,101,110} {001,011,101,201},{001,011,101,210},{001,011,110,201} {001,011,110,210},{001,011,201,210},{001,012,021,102} {001,012,021,120},{001,012,021,201},{001,012,021,210} {001,012,102,120},{001,012,102,201},{001,012,102,210} {001,012,120,201},{001,012,120,210},{001,012,201,210} {000,001,102,110},{000,001,101,110},{000,001,100,110} {000,001,110,201},{000,001,110,210}	1,2,3,4,5,6,7,8,9,10
17	{000,010,011,102},{000,010,011,120},{000,001,102,210} {000,001,101,210},{000,001,100,210},{000,001,201,210}	1,2,3,5,7,9,11,13,15,17
18	{000,010,011,100},{000,010,011,101},{000,010,011,110} {000,010,011,201},{000,010,011,210},{010,011,012,210}	1,2,3,5,8,12,17,23,30,38
19	{010,011,012,100},{010,011,012,101},{010,011,012,102} {010,011,012,110},{010,011,012,120},{010,011,012,201} {000,001,100,102},{000,001,101,102},{000,001,100,101} {000,001,102,201},{000,001,101,201},{000,001,100,201}	1,2,3,5,8,13,21,34,55,89
T20	{000,010,102,120}	1,2,4,10,26,66,172,457,1225,3311
T21	{000,010,102,110}	1,2,4,10,26,67,177,475,1287,3518
T22	{000,010,100,102}	1,2,4,10,26,68,187,523,1486,4290
T23	{000,010,101,102}	1,2,4,10,26,70,195,557,1619,4777
T24	{000,010,100,120}	1,2,4,10,26,70,197,567,1666,4988
T25	{000,010,100,110}	1,2,4,10,26,71,201,587,1756,5361
T26	{000,010,101,110}	1,2,4,10,26,71,201,588,1767,5438
27	{000,010,101,120},{000,010,110,120}	1,2,4,10,26,71,202,593,1785,5493
T28	{000,010,100,101}	1,2,4,10,26,73,217,678,2213,7521
T29	{000,010,102,210}	1,2,4,10,27,73,202,568,1612,4606
T30	{000,010,102,201}	1,2,4,10,27,73,203,577,1667,4881
T31	{000,010,120,201}	1,2,4,10,27,76,223,675,2091,6598
T32	{000,010,120,210}	1,2,4,10,27,76,223,677,2109,6717
T33	{000,010,110,201}	1,2,4,10,27,77,228,696,2175,6925
T34	{000,010,110,210}	1,2,4,10,27,77,228,698,2198,7092
T35	{000,010,100,201}	1,2,4,10,27,78,239,764,2532,8649
T36	{000,010,100,210}	1,2,4,10,27,78,239,766,2554,8809

Continuation of Table 5		
Class	B quadruple	$\{\|WA_n(B)\|\}_{n=1}^{10}$
T37	{000,010,101,201}	1,2,4,10,27,79,245,797,2696,9425
T38	{000,010,101,210}	1,2,4,10,27,79,245,799,2720,9614
T39	{000,010,201,210}	1,2,4,10,28,85,273,912,3139,11055
40	{000,012,021,101},{000,012,021,110}	1,2,4,3,0,0,0,0,0,0
T41	{000,012,101,110}	1,2,4,3,1,0,0,0,0,0
42	{000,012,100,110},{000,012,021,100},{000,012,021,102} {000,012,021,120},{000,012,021,201},{000,012,021,210}	1,2,4,4,0,0,0,0,0,0
43	{000,012,100,101},{000,012,102,110},{000,012,110,120} {000,012,110,201},{000,012,110,210}	1,2,4,4,1,0,0,0,0,0
44	{000,012,101,102},{000,012,101,120},{000,012,101,201} {000,012,101,210}	1,2,4,4,2,1,0,0,0,0
45	{000,012,100,102},{000,012,100,120},{000,012,100,201} {000,012,100,210}	1,2,4,5,1,0,0,0,0,0
46	{000,012,102,120},{000,012,102,201},{000,012,102,210} {000,012,120,201},{000,012,120,210},{000,012,201,210}	1,2,4,5,2,1,0,0,0,0
47	{000,011,102,120},{000,011,021,102},{000,011,021,120} {001,100,110,120},{001,021,100,120},{001,021,110,120} {001,021,100,110}	1,2,4,5,6,7,8,9,10,11
48	{000,011,100,102},{000,011,100,120},{011,012,021,100} {000,011,101,102},{000,011,102,110},{000,011,102,201} {000,011,102,210},{000,011,101,120},{000,011,110,120} {000,011,120,201},{000,011,120,210},{000,011,021,100} {000,011,021,101},{000,011,021,110},{000,011,021,201} {000,011,021,210},{001,100,102,120},{001,102,110,120} {001,100,101,120},{001,101,110,120},{001,100,102,110} {001,100,101,110},{001,021,101,120},{001,021,102,120} {001,021,100,102},{001,100,120,201},{001,100,120,210} {001,021,102,110},{001,021,100,101},{001,110,120,201} {001,110,120,210},{001,021,101,110},{001,100,110,201} {001,100,110,210},{001,021,120,201},{001,021,120,210} {001,021,100,210},{001,021,100,201},{001,021,110,210} {001,021,110,201}	1,2,4,6,8,10,12,14,16,18
49	{000,011,100,101},{000,011,100,110},{000,011,100,201} {000,011,100,210},{010,012,021,100},{010,012,021,101} {010,012,021,102},{010,012,021,110},{010,012,021,120} {010,012,021,201},{010,012,021,210},{011,012,021,101} {011,012,021,102},{011,012,021,110},{011,012,021,120} {011,012,021,201},{011,012,021,210},{000,011,101,110} {000,011,101,201},{000,011,101,210},{000,011,110,201} {000,011,110,210},{000,011,201,210},{001,021,201,210} {001,101,102,120},{001,101,102,110},{001,101,120,201} {001,101,120,210},{001,102,120,201},{001,102,120,210} {001,102,110,201},{001,100,102,210},{001,102,110,210} {001,100,101,210},{001,021,101,102},{001,101,110,201} {001,101,110,210},{001,021,102,210},{001,021,102,201} {001,021,101,201},{001,021,101,210},{001,100,201,210} {001,120,201,210},{001,110,201,210}	1,2,4,7,11,16,22,29,37,46
T50	{011,012,100,210}	1,2,4,7,12,19,28,39,52,67
51	{011,012,100,201},{001,100,101,102},{001,100,102,201} {001,100,101,201}	1,2,4,7,12,20,33,54,88,143
52	{011,012,100,101},{011,012,100,102},{011,012,100,110} {011,012,100,120}	1,2,4,7,13,23,41,72,126,219
T53	{011,012,201,210}	1,2,4,8,14,22,32,44,58,74
T54	{010,012,100,110}	1,2,4,8,14,23,36,55,83,125
55	{010,012,100,210},{010,012,110,210},{011,012,101,210} {011,012,102,210},{011,012,110,210},{011,012,120,210} {001,101,102,210},{001,102,201,210},{001,101,201,210}	1,2,4,8,15,26,42,64,93,130
56	{010,012,100,101},{010,012,100,102},{010,012,100,120} {010,012,100,201},{010,012,101,110},{010,012,102,110} {010,012,110,120},{010,012,110,201},{011,012,101,201} {011,012,102,201},{011,012,110,201},{011,012,120,201}	1,2,4,8,15,27,47,80,134,222
57	{010,012,101,210},{010,012,102,210},{010,012,120,210} {010,012,201,210}	1,2,4,8,16,31,57,99,163,256

Continuation of Table 5		
Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
58	{010,011,021,100},{010,011,021,101},{010,011,021,102} {010,011,021,110},{010,011,021,120},{010,011,021,201} {010,011,021,210},{010,012,101,102},{010,012,101,120} {010,012,101,201},{010,012,102,120},{010,012,102,201} {010,012,120,201},{011,012,101,102},{011,012,101,110} {011,012,101,120},{011,012,102,110},{011,012,102,120} {011,012,110,120},{001,101,102,201}	1,2,4,8,16,32,64,128,256,512
T59	{010,011,102,120}	1,2,4,9,20,45,100,221,484,1053
T60	{010,011,102,210}	1,2,4,9,21,50,119,281,656,1513
61	{010,011,102,201},{010,011,120,201}	1,2,4,9,21,50,120,289,697,1682
T62	{010,011,120,210}	1,2,4,9,21,50,121,297,737,1845
63	{010,011,100,102},{010,011,101,102},{010,011,102,110}	1,2,4,9,21,51,126,316,799,2034
64	{000,010,021,100},{000,010,021,101},{000,010,021,102} {000,010,021,110},{000,010,021,120},{000,010,021,201} {000,010,021,210}	1,2,4,9,21,51,127,323,835,2188
65	{010,011,100,120},{010,011,101,120},{010,011,110,120}	1,2,4,9,21,51,127,324,842,2225
T66	{010,011,201,210}	1,2,4,9,22,56,145,378,988,2585
67	{010,011,100,201},{010,011,101,201},{010,011,110,201}	1,2,4,9,22,57,154,429,1223,3550
68	{010,011,100,210},{010,011,101,210},{010,011,110,210}	1,2,4,9,22,57,155,439,1286,3875
69	{010,011,100,101},{010,011,100,110},{010,011,101,110}	1,2,4,9,22,58,164,494,1577,5311
70	{012,021,100,101},{012,021,100,110},{012,021,101,110}	1,2,5,10,17,26,37,50,65,82
T71	{012,100,101,110}	1,2,5,10,19,33,56,93,154,255
T72	{012,100,110,210}	1,2,5,11,21,35,53,75,101,131
73	{012,021,100,102},{012,021,100,120},{012,021,100,201} {012,021,100,210},{012,021,101,102},{012,021,101,120} {012,021,101,201},{012,021,101,210},{012,021,102,110} {012,021,110,120},{012,021,110,201},{012,021,110,210}	1,2,5,11,21,36,57,85,121,166
T74	{012,100,110,201}	1,2,5,11,21,36,58,90,137,207
T75	{000,021,102,120}	1,2,5,11,21,51,127,323,835,2188
76	{012,100,102,110},{012,100,110,120}	1,2,5,11,22,39,66,108,175,283
T77	{012,100,101,210}	1,2,5,11,22,40,67,105,156,222
T78	{012,101,110,210}	1,2,5,11,22,41,72,120,191,292
79	{012,100,101,201},{012,101,110,201}	1,2,5,11,22,41,73,126,213,355
T80	{011,021,102,120}	1,2,5,11,22,42,79,149,284,548
T81	{000,021,102,110}	1,2,5,11,22,52,128,324,836,2189
82	{012,100,101,102},{012,100,101,120}	1,2,5,11,23,45,85,156,281,499
83	{011,021,100,102},{011,021,100,120},{012,101,102,110} {012,101,110,120}	1,2,5,11,23,47,95,191,383,767
T84	{011,100,102,120}	1,2,5,11,25,55,121,263,569,1223
T85	{000,021,101,110}	1,2,5,11,25,59,144,361,924,2404
86	{000,021,101,120},{000,021,101,102},{000,021,110,120}	1,2,5,11,25,60,148,374,962,2511
87	{012,100,201,210},{012,110,201,210}	1,2,5,12,25,46,77,120,177,250
88	{000,021,102,201},{000,021,102,210},{000,021,100,102}	1,2,5,12,25,60,148,374,962,2511
89	{012,100,102,210},{012,100,120,210},{012,101,201,210} {012,102,110,210},{012,110,120,210}	1,2,5,12,26,51,92,155,247,376
90	{012,100,102,201},{012,100,120,201},{012,102,110,201} {012,110,120,201}	1,2,5,12,26,51,93,161,269,439
T91	{012,100,102,120}	1,2,5,12,27,56,110,207,378,675
92	{012,101,102,210},{012,101,120,210}	1,2,5,12,27,57,113,211,373,628
93	{011,102,120,201},{011,102,120,210},{011,021,101,102} {011,021,102,110},{011,021,102,201},{011,021,102,210} {012,101,102,201},{012,101,120,201},{012,102,110,120} {012,021,102,120},{012,021,102,201},{012,021,102,210} {012,021,120,201},{012,021,120,210},{012,021,201,210}	1,2,5,12,27,58,121,248,503,1014
94	{000,021,120,201},{000,021,120,210},{000,021,100,120}	1,2,5,12,27,65,160,404,1038,2707
95	{011,101,102,120},{011,102,110,120},{011,021,100,101} {011,021,100,110},{011,021,100,201},{011,021,100,210} {011,021,101,120},{011,021,110,120},{011,021,120,201} {011,021,120,210}	1,2,5,12,28,64,144,320,704,1536
T96	{012,101,102,120}	1,2,5,12,28,65,151,351,816,1897
97	{000,021,110,210},{000,021,110,201},{000,021,100,110}	1,2,5,12,28,67,164,411,1050,2726
T98	{011,100,102,210}	1,2,5,12,29,69,162,375,857,1936

Continuation of Table 5		
Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
99	{011,100,102,201},{011,100,120,201}	1,2,5,12,29,70,169,408,985,2378
T100	{011,100,120,210}	1,2,5,12,29,71,176,440,1108,2807
101	{011,100,101,102},{011,100,102,110}	1,2,5,12,30,75,190,483,1235,3167
102	{011,100,101,120},{011,100,110,120}	1,2,5,12,30,76,197,518,1383,3737
T103	{000,102,110,120}	1,2,5,12,30,77,200,528,1408,3791
T104	{000,101,102,120}	1,2,5,12,30,77,201,532,1424,3847
105	{000,021,101,201},{000,021,101,210},{000,021,100,101}	1,2,5,12,30,77,202,539,1458,3988
T106	{000,101,102,110}	1,2,5,12,31,81,216,583,1590,4372
T107	{000,101,110,120}	1,2,5,12,32,88,250,731,2183,6647
T108	{000,100,102,120}	1,2,5,13,31,80,207,542,1439,3854
109	{012,102,201,210},{012,120,201,210}	1,2,5,13,32,73,156,318,629,1224
T110	{011,102,201,210}	1,2,5,13,32,75,170,377,824,1783
T111	{000,102,120,201}	1,2,5,13,32,84,215,566,1494,3992
T112	{000,102,120,210}	1,2,5,13,32,84,217,575,1528,4107
113	{012,102,120,201},{012,102,120,210}	1,2,5,13,33,80,185,411,885,1862
114	{011,101,102,210},{011,102,110,210}	1,2,5,13,33,81,193,449,1025,2305
115	{011,101,102,201},{011,102,110,201}	1,2,5,13,33,82,201,489,1185,2866
T116	{011,120,201,210}	1,2,5,13,33,83,209,526,1319,3292
T117	{000,102,110,201}	1,2,5,13,33,87,228,609,1636,4437
T118	{000,100,102,110}	1,2,5,13,33,87,231,621,1686,4612
119	{000,021,100,201},{000,021,100,210},{000,021,201,210}	1,2,5,13,33,87,232,629,1724,4772
120	{011,101,120,201},{011,110,120,201}	1,2,5,13,34,89,233,610,1595,4161
121	{011,100,201,210},{011,101,102,110}	1,2,5,13,34,89,233,610,1597,4181
T122	{000,102,110,210}	1,2,5,13,34,90,240,645,1745,4750
123	{011,101,120,210},{011,110,120,210}	1,2,5,13,34,90,241,651,1769,4826
T124	{000,100,101,120}	1,2,5,13,35,100,293,880,2698,8409
125	{011,100,101,201},{011,100,110,201},{011,021,101,110} {011,021,101,201},{011,021,101,210},{011,021,110,201} {011,021,110,210},{011,021,201,210}	1,2,5,13,35,97,275,794,2327,6905
T126	{011,101,110,120}	1,2,5,13,35,97,275,795,2335,6953
127	{011,100,101,210},{011,100,110,210}	1,2,5,13,35,98,284,847,2589,8085
T128	{000,100,101,110}	1,2,5,13,35,99,290,871,2672,8355
T129	{000,101,102,210}	1,2,5,13,36,101,288,827,2389,6928
T130	{000,100,101,102}	1,2,5,13,36,104,308,934,2881,9014
T131	{000,101,110,201}	1,2,5,13,36,104,309,939,2905,9118
T132	{000,100,110,120}	1,2,5,13,36,104,309,943,2936,9298
T133	{000,101,120,201}	1,2,5,13,36,104,311,954,2987,9509
T134	{000,101,110,210}	1,2,5,13,36,104,311,959,3035,9824
T135	{000,101,120,210}	1,2,5,13,36,104,312,963,3045,9825
T136	{011,100,101,110}	1,2,5,13,36,106,330,1083,3734,13483
T137	{000,101,102,201}	1,2,5,13,37,107,321,979,3042,9573
T138	{000,110,120,201}	1,2,5,13,37,108,328,1018,3224,10368
T139	{000,110,120,210}	1,2,5,13,37,108,330,1034,3327,10922
T140	{000,100,102,210}	1,2,5,14,38,107,304,868,2494,7193
T141	{000,102,201,210}	1,2,5,14,38,108,301,854,2425,6932
T142	{000,100,102,201}	1,2,5,14,38,110,323,972,2969,9196
T143	{000,100,120,201}	1,2,5,14,40,120,369,1161,3714,12050
T144	{000,100,120,210}	1,2,5,14,40,120,371,1176,3806,12528
145	{011,101,201,210},{011,110,201,210}	1,2,5,14,41,123,375,1157,3603,11304
T146	{000,100,110,201}	1,2,5,14,41,125,392,1257,4102,13579
T147	{000,120,201,210}	1,2,5,14,41,126,398,1292,4275,14375
T148	{000,110,201,210}	1,2,5,14,42,131,421,1384,4634,15750
149	{010,021,100,101},{010,021,100,102},{010,021,100,110} {010,021,100,120},{010,021,100,201},{010,021,100,210} {010,021,101,102},{010,021,101,110},{010,021,101,120} {010,021,101,201},{010,021,101,210},{010,021,102,110} {010,021,102,120},{010,021,102,201},{010,021,102,210} {010,021,110,120},{010,021,110,201},{010,021,110,210} {010,021,120,201},{010,021,120,210},{010,021,201,210} {011,101,110,201}	1,2,5,14,42,132,429,1430,4862,16796
T150	{011,101,110,210}	1,2,5,14,42,133,440,1508,5320,19225

Continuation of Table 5		
Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
T151	{000,100,110,210}	1,2,5,14,42,133,440,1510,5347,19457
T152	{000,100,101,210}	1,2,5,14,42,134,447,1547,5518,20193
T153	{000,101,201,210}	1,2,5,14,43,139,468,1623,5762,20844
T154	{000,100,101,201}	1,2,5,14,43,141,486,1744,6466,24634
T155	{000,100,201,210}	1,2,5,15,48,162,565,2023,7389,27428
T156	{010,100,102,120}	1,2,5,15,49,167,581,2049,7301,26239
T157	{010,102,110,120}	1,2,5,15,49,167,582,2058,7357,26534
T158	{010,101,102,120}	1,2,5,15,49,167,583,2068,7423,26896
T159	{010,100,102,110}	1,2,5,15,49,167,583,2071,7455,27126
T160	{010,101,102,110}	1,2,5,15,49,168,593,2135,7797,28784
T161	{010,100,101,102}	1,2,5,15,49,170,614,2285,8700,33729
162	{010,100,101,120},{010,100,110,120}	1,2,5,15,49,170,615,2297,8796,34370
T163	{010,101,110,120}	1,2,5,15,49,170,616,2309,8895,35055
T164	{010,100,101,110}	1,2,5,15,49,170,618,2333,9093,36449
T165	{010,102,120,201}	1,2,5,15,50,174,614,2178,7758,27767
T166	{010,102,120,210}	1,2,5,15,50,174,616,2201,7919,28665
T167	{010,102,110,201}	1,2,5,15,50,174,616,2202,7933,28783
T168	{010,102,110,210}	1,2,5,15,50,174,617,2211,7983,29003
T169	{010,100,102,210}	1,2,5,15,50,176,638,2354,8789,33099
T170	{010,100,102,201}	1,2,5,15,50,176,639,2371,8953,34310
T171	{010,101,102,210}	1,2,5,15,50,177,649,2431,9230,35360
T172	{010,101,102,201}	1,2,5,15,50,177,651,2460,9489,37205
173	{010,100,120,201},{010,110,120,201}	1,2,5,15,50,178,662,2542,10006,40173
174	{010,100,110,201},{010,101,110,201}	1,2,5,15,50,178,663,2552,10071,40528
T175	{010,101,120,201}	1,2,5,15,50,178,663,2554,10100,40790
T176	{010,100,110,210}	1,2,5,15,50,178,664,2566,10195,41425
T177	{010,100,120,210}	1,2,5,15,50,178,664,2567,10209,41546
178	{010,101,120,210},{010,110,120,210}	1,2,5,15,50,178,664,2568,10225,41703
T179	{010,101,110,210}	1,2,5,15,50,178,665,2582,10351,42641
T180	{010,100,101,201}	1,2,5,15,50,180,688,2758,11493,49454
T181	{010,100,101,210}	1,2,5,15,50,180,689,2773,11637,50596
T182	{010,102,201,210}	1,2,5,15,51,184,679,2529,9474,35671
T183	{010,110,201,210}	1,2,5,15,51,187,718,2845,11547,47782
T184	{010,120,201,210}	1,2,5,15,51,187,719,2863,11727,49145
185	{010,100,201,210},{010,101,201,210}	1,2,5,15,51,189,744,3059,12991,56557
186	{021,100,102,120},{021,101,102,120},{021,102,110,120}	1,2,6,18,52,152,464,1486,4946,16916
T187	{021,100,102,110}	1,2,6,18,52,153,470,1508,5010,17079
T188	{021,101,102,110}	1,2,6,18,53,158,486,1550,5109,17298
T189	{021,100,101,110}	1,2,6,18,55,172,551,1806,6043,20588
190	{021,100,101,120},{021,101,110,120},{021,100,101,102} {021,100,110,120}	1,2,6,18,55,173,560,1858,6291,21657
191	{021,102,110,201},{021,102,110,210}	1,2,6,19,57,168,506,1585,5165,17382
192	{021,102,120,201},{021,102,120,210}	1,2,6,19,58,174,528,1649,5328,17764
193	{021,100,102,201},{021,100,102,210}	1,2,6,19,59,183,580,1893,6347,21741
194	{021,101,102,201},{021,101,102,210}	1,2,6,19,60,191,619,2048,6909,23704
195	{021,100,110,201},{021,100,110,210}	1,2,6,19,61,198,651,2171,7345,25194
196	{021,101,120,201},{021,101,120,210},{021,100,120,201} {021,100,120,210},{021,110,120,201},{021,110,120,210}	1,2,6,19,61,199,661,2234,7668,26674
197	{021,101,110,201},{021,101,110,210}	1,2,6,19,62,207,703,2420,8424,29602
T198	{100,102,110,120}	1,2,6,19,63,212,726,2521,8863,31489
T199	{100,101,102,120}	1,2,6,19,63,213,733,2558,9034,32228
T200	{101,102,110,120}	1,2,6,19,63,215,749,2650,9490,34318
201	{021,100,101,201},{021,100,101,210}	1,2,6,19,63,216,759,2717,9867,36244
T202	{100,101,102,110}	1,2,6,19,64,222,788,2842,10378,38266
T203	{100,101,110,120}	1,2,6,19,65,233,866,3308,12916,51334
T204	{021,102,201,210}	1,2,6,20,66,213,683,2211,7291,24552
205	{100,102,120,201},{102,110,120,201}	1,2,6,20,68,231,788,2711,9423,33091
206	{100,102,120,210},{101,102,120,201},{101,102,120,210} {102,110,120,210}	1,2,6,20,68,233,805,2807,9879,35073
T207	{100,102,110,201}	1,2,6,20,68,234,816,2882,10294,37124
T208	{100,102,110,210}	1,2,6,20,69,240,842,2979,10625,38177

Continuation of Table 5		
Class	B quadruple	$\{\lvert WA_n(B)\rvert\}_{n=1}^{10}$
T209	{021,110,201,210}	1,2,6,20,69,242,857,3056,10959,39493
T210	{101,102,110,210}	1,2,6,20,69,242,858,3068,11050,40052
211	{101,102,110,201},{021,120,201,210}	1,2,6,20,69,242,859,3080,11140,40596
T212	{021,100,201,210}	1,2,6,20,70,252,924,3432,12870,48620
T213	{100,101,102,210}	1,2,6,20,71,260,970,3662,13938,53364
214	{100,101,120,201},{101,110,120,201}	1,2,6,20,71,263,1007,3958,15881,64778
215	{100,101,110,201},{021,101,201,210}	1,2,6,20,71,264,1015,4002,16094,65758
T216	{100,101,120,210}	1,2,6,20,71,264,1018,4040,16402,67817
T217	{100,101,110,210}	1,2,6,20,71,265,1027,4097,16727,69600
T218	{101,110,120,210}	1,2,6,20,71,265,1029,4123,16943,71086
219	{100,101,102,201},{100,110,120,201}	1,2,6,20,72,272,1064,4272,17504,72896
T220	{100,110,120,210}	1,2,6,20,72,273,1076,4367,18137,76739
T221	{102,110,201,210}	1,2,6,21,74,258,897,3131,11007,39007
T222	{102,120,201,210}	1,2,6,21,75,265,927,3230,11268,39486
T223	{100,102,201,210}	1,2,6,21,76,277,1016,3756,13998,52554
T224	{101,102,201,210}	1,2,6,21,77,287,1079,4082,15522,59280
T225	{101,120,201,210}	1,2,6,21,78,301,1203,4955,20888,89611
226	{101,110,201,210},{100,110,201,210}	1,2,6,21,79,311,1265,5275,22431,96900
227	{100,120,201,210},{110,120,201,210}	1,2,6,21,79,313,1290,5475,23764,105001
T228	{100,101,201,210}	1,2,6,21,80,322,1347,5798,25512,114236
End of Table 5		