ARITHMETICAL STRUCTURES ON COCONUT TREES

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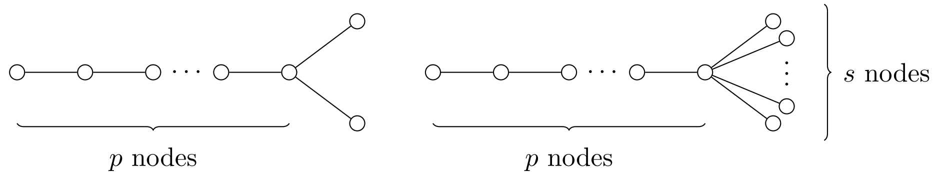
Received: 6/26/24, Revised: 4/8/25, Accepted: 8/20/25, Published: 9/17/25

Abstract

If G is a finite connected graph, then an arithmetical structure on G is a pair of vectors $(\mathbf{d}, \mathbf{r})$ with positive integer entries such that $(\operatorname{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$, where A is the adjacency matrix of G and the entries of $\mathbf{r}$ have no common factor other than 1. In this paper, we generalize the result of Archer, Bishop, Diaz-Lopez, García Puente, Glass, and Louwsma on enumerating arithmetical structures on bidents (also called coconut tree graphs $\mathrm{CT}(p, 2)$) to any coconut tree graphs $\mathrm{CT}(p, s)$ which consists of a path on $p > 0$ vertices to which we append $s > 0$ leaves to the right most vertex on the path. We also give a characterization of smooth arithmetical structures on coconut trees when given number assignments to the leaf nodes.

DOI: 10.5281/zenodo.17144351

[1]A. Diaz-Lopez was supported by NSF DMS-2211379.
[2]P. E. Harris was supported through a Karen Uhlenbeck EDGE Fellowship.



(a) $\mathrm{CT}(p,2)$ (b) $\mathrm{CT}(p,s)$

Figure 1: Examples of coconut tree graphs.

1. Introduction

An arithmetical structure on a graph is an assignment of weights to the vertices of the graph with positive integers satisfying: the weight at a vertex divides the sum of its neighbors' weights (with multiplicity), and the weights do not share any common factor other than 1. Alternatively, they can be defined as a pair of vectors that satisfy a collection of linear equations that are equivalent to the divisibility conditions (Definition 1). Arithmetical structures arose out of the study of the intersection of degenerating curves in algebraic geometry [11]. Recently, they have been studied from a combinatorial point of view [1, 3, 5, 8, 10, 12, 13] and from an algebraic point of view [2, 6, 14].

Lorenzini established that finite simple graphs have finitely many arithmetical structures [11, Lemma 1.6]. In light of Lorenzini's result, it is of interest to enumerate arithmetical structures on families of graphs. Throughout, we let $\mathbf{Arith}(G)$ denote the set of arithmetical structures on a graph G and $|\mathbf{Arith}(G)|$ denotes its cardinality. In [3], Braun, Corrales, Corry, García Puente, Glass, Kaplan, Martin, Musiker, and Valencia established that if $\mathcal{P}_{n+1}$ is the path graph on $n+1$ vertices, then $|\mathbf{Arith}(\mathcal{P}_{n+1})| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number [7, A000108], and if $\mathcal{C}_n$ is the cycle graph on n vertices, then $|\mathbf{Arith}(\mathcal{C}_n)| = \binom{2n-1}{n-1} = (2n-1)C_{n-1}$. Some partial enumeration results are known for bidents [1], paths with a double edge [8], and E_n graphs [13]. For complete graphs, arithmetical structures are in bijection with Egyptian fractions summing to 1 [7, A002967]. In [10], Keyes and Reiter provide a (very large) upper bound on the number of arithmetical structures on a graph based on the number of edges of the graph.

In this work, we enumerate arithmetical structures on the coconut tree graph $\mathrm{CT}(p,s)$, which consists of a path on $p > 0$ vertices to which we append $s > 0$ leaves to the rightmost vertex on the path. Figures 1a and 1b illustrate $\mathrm{CT}(p,2)$ and $\mathrm{CT}(p,s)$, respectively. We refer to the vertex to which we append leaves as the *central* vertex of the coconut tree. The coconut tree graph $\mathrm{CT}(p,2)$ is also referred to as a bident, and Archer et al. studied arithmetical structures on bidents by counting a smaller subset of arithmetical structures which they called smooth

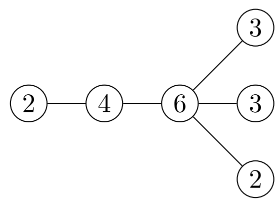


Figure 2: A smooth arithmetical structure on $\mathrm{CT}(3,3)$.

arithmetical structures [1]. In this context, an arithmetical structure is said to be *smooth* if, for each noncentral vertex, the sum of the weights of its neighbors is k times its weight for some k strictly greater than 1 (here k can vary from vertex to vertex). This condition guarantees that the structure cannot be obtained from a smaller structure by a process called subdivision. The technical definitions of these terms are given in Definitions 3 and 7. Throughout, we will let $\textbf{SArith}(G)$ denote the set of smooth arithmetical structures on a graph G. See Figure 2 for an example of a smooth arithmetical structure on $\mathrm{CT}(3,3)$.

Our main result generalizes the work of Archer et al. [1], and enumerates arithmetical structures on coconut trees $\mathrm{CT}(p,s)$, for all $p \geq 1$ and $s \geq 2$. Let $B(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{n}$ be a ballot number [7, A009766], which were introduced by Carlitz [4]. We can now state our main result, which reduces the problem of counting arithmetical structures on coconut trees to counting smooth arithmetical structures.

Theorem 1. *If $p \geq 1, s \geq 2$, then the number of arithmetical structures for the coconut tree $\mathrm{CT}(p,s)$ is given by*

$$|\textbf{\textit{Arith}}(\mathrm{CT}(p,s))| = \sum_{j=0}^{s} \binom{s}{j} A(p+s-j, s-j), \tag{1}$$

where

$$A(x,0) = C_{x-1} \text{ for } x \geq 1,$$
$$A(x,1) = C_{x-1} - C_{x-2} \text{ for } x \geq 2, \text{ and}$$
$$A(x,y) = \sum_{i=y+1}^{x} B(x-y-1, x-i)|\textbf{\textit{SArith}}(\mathrm{CT}(i-y,y))| \text{ for } x \geq 3, y \geq 2.$$

This paper is organized as follows. In Section 2, we give the necessary background on arithmetical structures on graphs, including the definition of smooth arithmetical structures. In Section 3, we describe the smoothing and subdividing operations needed in order to count arithmetical structures on coconut tree graphs. In Section 4, we prove Theorem 1 counting arithmetical structures on coconut trees by extending the definition of smooth arithmetical structures to coconut trees with $s \geq 2$. We end with Section 5 by presenting a few directions for further study.

Remark 1. We point the interested reader to GitHub [9] which contains code whose inputs are a specific graph G and a maximum value m, and whose output is the set of arithmetical structures on G with values less than or equal to m.

2. Background on Smooth Arithmetical Structures

In this section, we provide the necessary definitions and notation to make our approach precise. Throughout our work $G = (V, E)$ is a simple connected finite graph with vertex set V and edge set E. The vertices u and v are *adjacent* if there exists an edge between them. We denote the set of vertices of G by $V = \{v_1, v_2, \ldots, v_n\}$ and the *adjacency matrix* $A = (a_{i,j})$ of G is the square matrix of dimension $|V| \times |V|$ defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that A is symmetric.

Definition 1. Let G be a graph. An *arithmetical structure* on G is defined as a pair of vectors $(\mathbf{d}, \mathbf{r}) \in \mathbb{N}^{|V|} \times \mathbb{N}^{|V|}$ that satisfy

$$(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}, \tag{2}$$

the entries of $\mathbf{r}$ have no common factor other than 1, and where $\mathrm{diag}(\mathbf{d})$ is the diagonal matrix with entries given by the vector $\mathbf{d}$ and A is the adjacency matrix of G.

Equivalently, an arithmetical structure on G can be defined as an assignment of numbers to the vertices of G such that:

1. The assigned number of a vertex divides the sum of its neighbors' assigned numbers.
2. The greatest common divisor of all the assigned numbers is 1.

Throughout, we refer to most arithmetical structures by their $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ vector as it represents the assignment of values to the vertices of the graph G. Equation (2) means that the entries of the vector $\mathbf{d}$ measure by what factor the assigned number of each vertex divides the sum of its neighbors. Hence, $\mathbf{r}$ completely determines $\mathbf{d}$. Conversely, if $\mathbf{d}$ satisfies Equation (2) for some vector $\mathbf{r}$ with positive coefficients, Lorenzini [11, Proposition 1.1] showed that the matrix $(\mathrm{diag}(\mathbf{d}) - A)$ has rank $|V| - 1$, hence its kernel is one-dimensional and there is a unique vector $\mathbf{r}$

such that $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure (since the entries of $\mathbf{r}$ must be positive and have 1 as their only common factor). We do remark that just having a $\mathbf{d}$ vector that satisfies Equation (2) does not automatically imply that there is a vector in the kernel of $(\mathrm{diag}(\mathbf{d}) - A)$ with strictly positive coefficients [5, Remark 3.11].

We now formally define coconut tree graphs.

Definition 2. A coconut tree $\mathrm{CT}(p, s)$ for $p > 0$ and $s > 0$ is a path graph $\mathcal{P}_p$ with s leaf vertices at one of the ends. Note that $\mathrm{CT}(p, 1) = \mathcal{P}_{p+1}$ and $\mathrm{CT}(p, 0) = \mathcal{P}_p$.

Given a coconut tree graph $\mathrm{CT}(p, s)$, we label the path portion of the graph by v_1 to v_p, and denote the leaf vertices as $v_{\ell_1}, \ldots, v_{\ell_s}$, such that v_p is adjacent to all of the leaf vertices. This is illustrated in Figure 3. For vectors $\mathbf{r}$ and $\mathbf{d}$, we write $\mathbf{r} = (r_1, \ldots, r_p, r_{\ell_1}, \ldots, r_{\ell_s})$ and $\mathbf{d} = (d_1, \ldots, d_p, d_{\ell_1}, \ldots, d_{\ell_s})$, where each subscript i corresponds to the vertex v_i. Note that this differs slightly from the notation used in Archer et al. in [1].

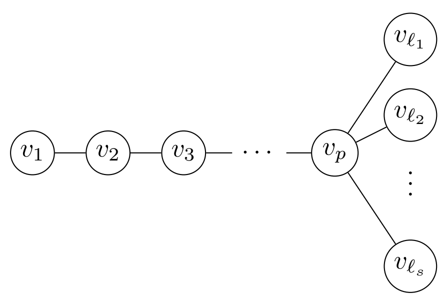


Figure 3: The coconut tree $\mathrm{CT}(p, s)$.

Definition 3. An arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(p, s)$ is *smooth* if

$$d_1, \ldots, d_{p-1}, d_{\ell_1}, \ldots, d_{\ell_s} \geq 2.$$

The set of smooth arithmetical structures on $\mathrm{CT}(p, s)$ is denoted $\mathbf{SArith(}\mathrm{CT}(p, s)\mathbf{)}$.

Note that the above definition does not place any restrictions on d_p. In the case of $\mathrm{CT}(p, 2)$, Archer et al. [1] proved that d_p must be 1 for smooth arithmetical structures. For $s \geq 3$, this result no longer holds. For example, consider the smooth arithmetical structure on $\mathrm{CT}(3, 3)$ presented in Figure 2. Observe that $\mathbf{r} = (r_1, r_2, r_3, r_{\ell_1}, r_{\ell_2}, r_{\ell_3}) = (2, 4, 6, 3, 3, 2)$ and $\mathbf{d} = (2, 2, 2, 2, 2, 3)$.

Lemma 1. *Let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathrm{CT}(p, s)$. The following conditions are equivalent:*

1. $d_i \geq 2$ *for* $1 \leq i \leq p - 1$;
2. $0 < r_2 - r_1 \leq \cdots \leq r_{p-1} - r_{p-2} \leq r_p - r_{p-1}$;

3. $r_1 < r_2 < \cdots < r_p$.

Proof. The proof is identical to Lemma 2.1 of [1] as the proof does not refer to any leaf vertices. □

Lemma 2. *An arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on* $\mathrm{CT}(p, s)$ *is smooth if and only if* $r_{\ell_j} < r_p$ *for all* $1 \leq j \leq s$ *and* $r_1 < r_2 < \cdots < r_p$.

Proof. ($\Rightarrow$) Let $(\mathbf{d}, \mathbf{r})$ be a smooth structure on $\mathrm{CT}(p, s)$. By the definition of smoothness, $d_i \geq 2$ for all $1 \leq i \leq p-1$ and $d_{\ell_j} \geq 2$ for all $1 \leq j \leq s$. Since $d_{\ell_j} r_{\ell_j} = r_p$, the condition $d_{\ell_j} \geq 2$ implies $r_{\ell_j} < r_p$ for $1 \leq j \leq s$, and by Lemma 1 item (3), $r_1 < r_2 < \cdots < r_p$.

($\Leftarrow$) Let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathrm{CT}(p, s)$, such that $r_{\ell_j} < r_p$ for all $1 \leq j \leq s$, and $r_1 < r_2 < \cdots < r_p$. By the divisibility condition, $d_{\ell_j} r_{\ell_j} = r_p$ for $1 \leq j \leq s$, and since $r_{\ell_j} < r_p$, this gives $d_{\ell_j} \geq 2$. By Lemma 1, $r_1 < r_2 < \cdots < r_p$ is equivalent to $d_i \geq 2$ for $1 \leq i \leq p-1$. Therefore, $d_i \geq 2$ for all $1 \leq i \leq p-1$ and $d_{\ell_j} \geq 2$ for all $1 \leq j \leq s$. Hence, $(\mathbf{d}, \mathbf{r})$ is a smooth structure. □

Lemma 3. *Let* $(\mathbf{d}, \mathbf{r})$ *be a smooth arithmetical structure on* $\mathrm{CT}(p, s)$ *with* $\mathbf{r} = (r_1, \ldots, r_p, r_{\ell_1}, \ldots, r_{\ell_s})$*, then* $\gcd(r_{\ell_1}, \ldots, r_{\ell_s}) = 1$.

Proof. Denote $g = \gcd(r_{\ell_1}, \ldots, r_{\ell_s})$. Since $r_{\ell_j} \mid r_p$ for all $1 \leq j \leq s$, this implies $g \mid r_p$. Since $r_{p-1} = d_p r_p - \sum_{j=1}^{s} r_{\ell_j}$, we have $g \mid r_{p-1}$. Consider the previous argument as the base case for induction. By induction, assume that $g \mid r_j$ for all $i \leq j \leq p$. Now we want to show that $g \mid r_{i-1}$. Note that $d_i = (r_{i+1} + r_{i-1})/r_i$ which simplifies to $r_{i-1} = d_i r_i - r_{i+1}$ (where we take r_{p+1} to be 0). By the induction hypothesis $g \mid r_{i+1}$ and $g \mid r_i$, which implies that $g \mid r_{i-1}$. Thus, $g \mid r_i$ for all $1 \leq i \leq p$. Thus, g divides every label on the graph $\mathrm{CT}(p, s)$, implying that $g = 1$. □

3. Smoothing and Subdivision

In this section, we present two operations that have proven useful in the enumeration of arithmetical structures on paths and cycles [3], bidents [1], E_n-graphs [13], as well as other works involving arithmetical structures [5, 6, 8, 10, 14].

3.1. Smoothing Arithmetical Structures on Coconut Trees

The process we describe for smoothing arithmetical structures on $\mathrm{CT}(p, s)$ is similar to what was done in [1] for smoothing arithmetical structures on $\mathrm{CT}(p, 2)$. Before stating the next definition we remark that in graph theory the use of the phrase "smoothing a vertex" refers to the replacement of a degree two vertex by an edge

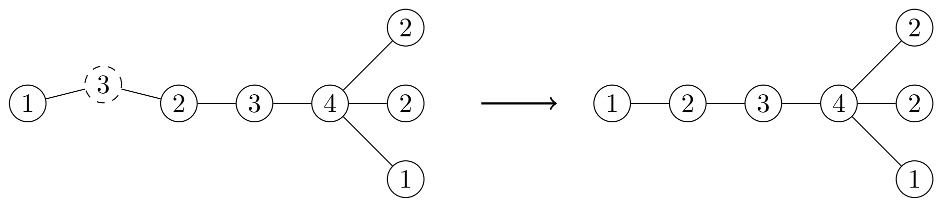


Figure 4: Smoothing the **r**-structure $(1,3,2,3,4,2,2,1)$ on $\mathrm{CT}(5,3)$ at v_2 (which we highlight as a dashed vertex) to get $\mathbf{r}'$-structure $(1,2,3,4,2,2,1)$ on $\mathrm{CT}(4,3)$.

connecting the two neighbors. We use the same naming convention for this operation, but stress that it does not necessarily imply "smoothness" in the sense of arithmetical structures, as divisibility conditions are generally not satisfied if you smooth at a vertex v with $d_v \neq 1$.

Definition 4. Let $p, s \in \mathbb{N}$. For $2 \leq i \leq p-1$, a *smoothing at the vertex* v_i *of degree* 2 *when* $d_i = 1$ is defined as an operation that takes an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(p, s)$ and returns an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ on $\mathrm{CT}(p-1, s)$, where the components of the vectors $\mathbf{d}'$, $\mathbf{r}'$ are as follows:

$$r'_j = \begin{cases} r_j & j \in \{1, 2, \ldots, i-1, \ell_1, \ell_2, \ldots, \ell_s\} \\ r_{j+1} & j \in \{i, i+1, \ldots, p-1\} \end{cases}$$

and

$$d'_j = \begin{cases} d_j & j \in \{1, \ldots, i-2, \ell_1, \ell_2, \ldots, \ell_s\} \\ d_j - 1 & j = i-1 \\ d_{j+1} - 1 & j = i \\ d_{j+1} & j \in \{i+1, \ldots, p-1\}. \end{cases}$$

The requirement that $d_i = 1$ is present to ensure that the resulting vectors $(\mathbf{d}', \mathbf{r}')$ form a structure on $\mathrm{CT}(p-1, s)$ as shown in Proposition 1. Before we present this result, we illustrate Definition 4.

Example 1. Consider the coconut tree $\mathrm{CT}(5, 3)$ with

$$\mathbf{d} = (3, 1, 3, 2, 2, 2, 2, 4) \text{ and } \mathbf{r} = (1, 3, 2, 3, 4, 2, 2, 1).$$

Since $d_2 = 1$, this arithmetical structure is not smooth. We can smooth at v_2 by removing v_2 and connecting v_1 to v_3, as illustrated in the right subfigure in Figure 4. Note that the removal of v_2 yields an arithmetical structure whose entries of the **d**-vector for v_1 and v_3 are both reduced by 1. Namely, the resulting arithmetical structure is

$$\mathbf{d}' = (2, 2, 2, 2, 2, 2, 4) \text{ and } \mathbf{r}' = (1, 2, 3, 4, 2, 2, 1),$$

which is a smooth arithmetical structure on $\mathrm{CT}(4, 3)$.

Proposition 1. *Let $p \geq 2$ and $s \geq 1$ be integers, and let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on* $\mathrm{CT}(p, s)$*. If $d_i = 1$ for some $1 \leq i \leq p-1$, then $(\mathbf{d}', \mathbf{r}')$ resulting from smoothing vertex v_i is a valid arithmetical structure on* $\mathrm{CT}(p-1, s)$*.*

Proof. There are two cases to consider:

- For $j < i-1$ or $j > i+1$, the neighbors of v_j are unchanged, so the divisibility condition still holds for $\mathbf{r}'$.
- For $j = i-1$ or $j = i+1$ consider the vertex v_j. Note that $d_i = 1$ implies $r_i = r_{i-1} + r_{i+1}$, and since $r_{i-1} | r_{i-2} + r_i$, we can substitute to get $r_{i-1} | r_{i-2} + r_{i-1} + r_{i+1}$. Now note that as $r_{i-1} | r_{i-1}$, then it must be that $r_{i-1} | r_{i-2} + r_{i+1}$. An analogous argument shows that $r_{i+1} | r_{i-1} + r_{i+2}$. Thus, ensuring that removing r_i preserves the divisibility condition on r_{i-1} and r_{i+1}.

Therefore $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathrm{CT}(p-1, s)$, as claimed. □

We now give the definition for smoothing at a degree one vertex. Note that the smoothing process on a degree one vertex removes either v_1 or one of the leaves.

Definition 5. Let $p, s \in \mathbb{N}$. A *smoothing at the vertex v_i of degree* 1 (for $i = 1$ or $i \in \{\ell_1, \ldots, \ell_s\}$) *when* $d_i = 1$ is defined as an operation that takes an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(p, s)$ and returns an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ on $\mathrm{CT}(p-1, s)$ or $\mathrm{CT}(p, s-1)$, where the components of the vectors $\mathbf{d}'$, $\mathbf{r}'$ are as follows. For v_1, we have

$$r'_j = \begin{cases} r_{j+1} & \text{if } j \in \{1, 2, \ldots p-1\} \\ r_j & \text{if } j \in \{\ell_1, \ldots, \ell_s\} \end{cases}$$

and

$$d'_j = \begin{cases} d_2 - 1 & \text{if } j = 1 \\ d_{j+1} & \text{if } j \in \{2, 3, 4, \ldots, p-1\} \\ d_j & \text{if } j \in \{\ell_1, \ldots, \ell_s\}, \end{cases}$$

and for a leaf vertex v_{ℓ_i}, we have

$$r'_j = \begin{cases} r_j & \text{if } j \in \{1, 2, \ldots, p, \ell_1, \ldots, \ell_{i-1}\} \\ r_{j+1} & \text{if } j \in \ell_i, \ell_{i+1}, \ldots, \ell_{s-1}\} \end{cases}$$

and

$$d'_j = \begin{cases} d_j & \text{if } j \in \{1, 2, \ldots, p-1, \ell_1, \ldots, \ell_{i-1}\} \\ d_j - 1 & \text{if } j = p \\ d_{j+1} & \text{if } j \in \{\ell_i, \ell_{i+1} \ldots, \ell_{s-1}\}. \end{cases}$$

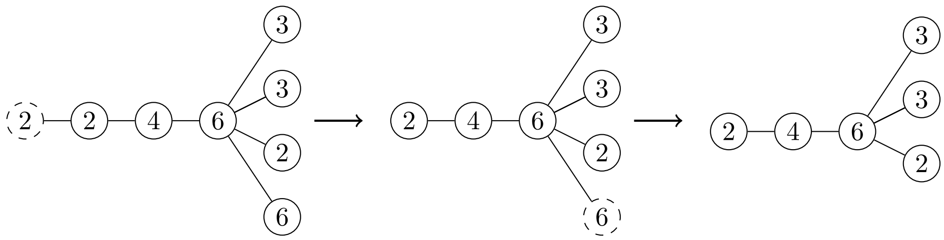


Figure 5: Smoothing the **r**-structure $(2, 2, 4, 6, 3, 3, 2, 6)$ on $\mathrm{CT}(4, 4)$ at vertex v_1 followed by smoothing at vertex v_{ℓ_4}. The right-most figure is the result of these smoothing operations.

The following example illustrates the smoothing process when $d_1 = 1$ or $d_{\ell_i} = 1$.

Example 2. Consider the arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(4, 4)$ with $\mathbf{d} = (1, 3, 2, 3, 2, 2, 3, 1)$ and $\mathbf{r} = (2, 2, 4, 6, 3, 3, 2, 6)$. We can first remove r_1 and relabel all of the path vertices to $r'_i = r_{i+1}$, reducing d_2 by 1. Then remove r_{ℓ_4}, which reduces d_4 (or d'_3) by 1, no relabeling is required for this removal. This gives $\mathbf{r}' = (2, 4, 6, 3, 3, 2)$ and $\mathbf{d}' = (2, 2, 2, 2, 2, 3)$, which is a smooth arithmetical structure on $\mathrm{CT}(3, 3)$, illustrated on the right in Figure 5.

Proposition 2. *Let $p, s \in \mathbb{N}$, and let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathrm{CT}(p, s)$. If, for some $i \in \{1, \ell_1, \ldots, \ell_s\}$, we have $d_i = 1$, then $(\mathbf{d}', \mathbf{r}')$ resulting from smoothing at vertex v_i is a valid arithmetical structure on $\mathrm{CT}(p-1, s)$ if $i = 1$ and $\mathrm{CT}(p, s-1)$ if $i \in \{\ell_1, \ldots, \ell_s\}$.*

Proof. There are two cases to consider.
`Case 1:` $d_1 = 1$. In this case, $r_1 = r_2$. We remove v_1 and relabel the vertices so that $r'_j = r_{j+1}$ for $j \in \{1, 2, \ldots, p-1\}$. Since $r_2 | r_1 + r_3$ and $r_1 = r_2$, it follows that $r_2 | r_3$, which implies that $d'_1 = d_2 - 1$. All other divisibility conditions remain unchanged, hence the result is a valid arithmetical structure.
`Case 2:` $d_{\ell_j} = 1$. In this case, $r_{\ell_j} = r_p$. We remove vertex v_{ℓ_j}. Since $r_{\ell_j} \equiv 0 \mod r_p$, subtracting r_{ℓ_j} does not change the divisibility condition at r_p. All other divisibility conditions remain unchanged, hence the result is a valid arithmetical structure. We also have that $d'_p = d_p - 1$.

Therefore, $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathrm{CT}(p-1, s)$ in the first case, and on $\mathrm{CT}(p, s-1)$ in second case, as claimed. □

We use the following concept of ancestor as in [1, 8].

Definition 6. Fix $p, s \in \mathbb{N}$. Let $1 \le q \le p$ and $1 \le t \le s$. An arithmetical structure $(\mathbf{d}', \mathbf{r}')$ on $\mathrm{CT}(q, t)$ is called an *ancestor* if it is obtained from a sequence of smoothing operations on an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(p, s)$. We call $(\mathbf{d}, \mathbf{r})$ a *descendant* of $(\mathbf{d}', \mathbf{r}')$ if and only if $(\mathbf{d}', \mathbf{r}')$ is an ancestor of $(\mathbf{d}, \mathbf{r})$.

Lemma 4. *Every arithmetical structure on* $\mathrm{CT}(p,s)$ *with* $d_{\ell_1},\ldots,d_{\ell_s} \geq 2$ *has a unique smooth arithmetical structure on* $\mathrm{CT}(q,s)$ *as an ancestor for some* q *satisfying* $1 \leq q \leq p$.

Proof. The proof is analogous to that of Lemma 2.6 in [1] and so we omit it. $\square$

3.2. Subdividing Arithmetical Structures on Coconut Trees

Recall that a smoothing operation can remove a vertex of degree 2 or less if its associated d-value is 1, which can only happen if the r-value of the vertex is equal to the sum of its neighbors. Then, a subdivision operation can be thought of as an inverse of the smoothing operation, as it always constructs a new vertex with an r-label that is equal to the sum of its neighbors. Subdivisions provide the foundation for enumerating arithmetical structures on coconut trees, as they reduce the problem to counting the number of smooth arithmetical structures on coconut trees.

Definition 7. Let $p, s \in \mathbb{N}$. A *subdivision at the vertex* v_i *for* $i \in \{1,2,\ldots,p\}$ (or "at position i") is defined as an operation that takes an arithmetical structure $(\mathbf{d},\mathbf{r})$ on $\mathrm{CT}(p,s)$ and returns an arithmetical structure $(\mathbf{d}',\mathbf{r}')$ on $\mathrm{CT}(p+1,s)$, where the components of the vectors $\mathbf{d}'$ and $\mathbf{r}'$ are as follows. If $i > 1$, then

$$r'_j = \begin{cases} r_j & \text{if } j \in \{1,\ldots,i-1,\, \ell_1,\ldots,\ell_s\} \\ r_{j-1} + r_j & \text{if } j = i \\ r_{j-1} & \text{if } j \in \{i+1, i+2, \ldots, p, p+1\} \end{cases}$$

and

$$d'_j = \begin{cases} d_j & \text{if } j \in \{1,\ldots,i-2, \ell_1,\ldots,\ell_s\} \\ d_j + 1 & \text{if } j = i-1 \\ 1 & \text{if } j = i \\ d_{j-1} + 1 & \text{if } j = i+1 \\ d_{j-1} & \text{if } j \in \{i+2,\ldots,p,p+1\}. \end{cases}$$

and if $i = 1$, then

$$r'_j = \begin{cases} r_1 & \text{if } j = 1 \\ r_{j-1} & \text{if } j \in \{2,3,\ldots,p+1\} \\ r_j & \text{if } j \in \{\ell_1,\ldots,\ell_s\} \end{cases}$$

and

$$d'_j = \begin{cases} 1 & \text{if } j = 1 \\ d_{j-1} + 1 & \text{if } j = 2 \\ d_{j-1} & \text{if } j \in \{3,4,\ldots,p,p+1\} \\ d_j & \text{if } j \in \{\ell_1,\ldots,\ell_s\}. \end{cases}$$

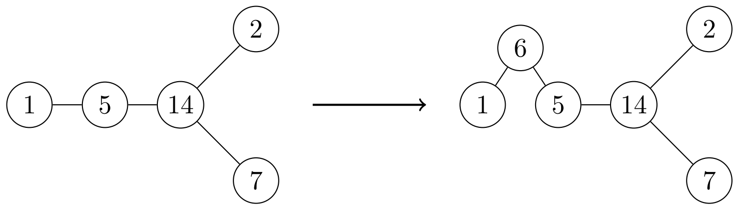


Figure 6: Subdividing at position 2 on an arithmetical structure on $\mathrm{CT}(3,2)$.

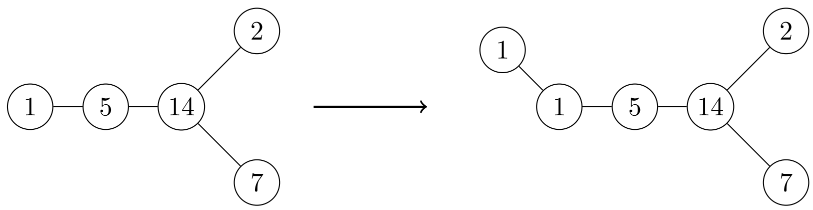


Figure 7: Subdividing at position 1 on an arithmetical structure on $\mathrm{CT}(3,2)$.

In short, a subdivision at position i takes the sum of r_{i-1} and r_i and assigns that value to a new vertex r'_i. We illustrate the definition through the following example.

Example 3. In Figure 6, we consider an arithmetical structure on $\mathrm{CT}(3,2)$ and subdivide at position 2 to get the structure on the right-hand side of the figure. Note the result is an arithmetical structure on $\mathrm{CT}(4,2)$. In Figure 7, we consider the same arithmetical structure on $\mathrm{CT}(3,2)$, and subdivide at position 1. This takes the beginning vertex with labeling 1 and assigns the same value 1 to a new vertex at the beginning of the path. Note the result is an arithmetical structure on $\mathrm{CT}(4,2)$.

Next, we prove that subdivision results in a valid arithmetical structure.

Proposition 3. *Let $p,s \in \mathbb{N}$, and let $(\mathbf{d},\mathbf{r})$ be an arithmetical structure on $\mathrm{CT}(p,s)$. We have that $(\mathbf{d}',\mathbf{r}')$ resulting from subdividing vertex $i \in \{1,\ldots,p\}$ is a valid arithmetical structure on $\mathrm{CT}(p+1,s)$.*

Proof. Note that the gcd condition is unchanged by adding a new vertex, so we only need to check divisibility. There are two cases:

- For $i=1$, note that $r'_1 = r'_2 = r_1$, so $r'_1 \mid r'_2$. Since $r_1 \mid r_2$ and $r'_3 = r_2$, then $r'_2 \mid r'_1 + r'_3$. For all other vertices, the divisibility condition does not change, so $(\mathbf{d}',\mathbf{r}')$ is an arithmetical structure on $\mathrm{CT}(p+1,s)$.
- For $i>1$, the only divisibility conditions that have changed are related to the new vertex and its two adjacent vertices. Note that $r_{i-1} \mid r_{i-2} + r_i$, hence

$r_{i-1} \mid r_{i-2} + (r_{i-1} + r_i)$, which is equivalent to $r'_{i-1} \mid r'_{i-2} + r'_i$. Regarding r'_i, it divides the sum of the labels on its two neighbors as it is defined as exactly that sum, that is, $r'_i = r_{i-1} + r_i = r'_{i-1} + r'_{i+1}$. Finally, a similar argument to the one for r'_{i-1} shows that r'_{i+1} (which equals r_i) divides the sum of its neighbors as the only change in the sum of its neighbors is the addition of r_i. So $(\mathbf{d}', \mathbf{r}')$ is a valid arithmetical structure on $\mathrm{CT}(p+1, s)$. □

4. Counting Arithmetical Structures on Coconut Trees

We now focus our attention on enumerating arithmetical structures on coconut trees. To begin, we define a subdivision sequence.

Definition 8. Let $(\mathbf{d}^0, \mathbf{r}^0)$ be an arithmetical structure on $\mathrm{CT}(p, s)$, with $p, s \in \mathbb{N}$. A sequence of positive integers $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is a *valid subdivision sequence* if its entries satisfy $1 \leq b_i \leq p + i - 1$ for $i \in \{1, 2, \ldots, k\}$.

The arithmetical structure $\mathbf{Sub}((\mathbf{d}^0, \mathbf{r}^0),\ \mathbf{b})$ on $\mathrm{CT}(p, s)$ is inductively defined as follows. Let $(\mathbf{d}^i, \mathbf{r}^i)$ be the arithmetical structure on $\mathrm{CT}(p+i, s)$ obtained from the arithmetical structure $(\mathbf{d}^{i-1}, \mathbf{r}^{i-1})$ on $\mathrm{CT}(p+i-1, s)$ by subdividing at the vertex v_{b_i}. Then, let

$$\mathbf{Sub}((\mathbf{d}^0, \mathbf{r}^0),\ \mathbf{b}) \coloneqq (\mathbf{d}^k, \mathbf{r}^k) \text{ on } \mathrm{CT}(p+k, s).$$

Example 4. Consider the arithmetical structure $(\mathbf{d}^0, \mathbf{r}^0)$ on $\mathrm{CT}(8, 3)$ with $\mathbf{d}^0 = (2, 2, 2, 2, 2, 2, 2, 2, 8, 2, 2)$ and $\mathbf{r}^0 = (1, 2, 3, 4, 5, 6, 7, 8, 1, 4, 4)$. Let $\mathbf{b} = (3, 4, 4, 7)$. Note that $3 \leq 8, 4 \leq 9, 4 \leq 10$, and $7 \leq 11$, so $\mathbf{b}$ is a valid subdivision sequence. We now construct $(\mathbf{d}^i, \mathbf{r}^i)$ for $i = 1, 2, 3, 4$,

$$\begin{aligned}
\mathbf{d}^1 &= (2, 3, \mathbf{1}, 3, 2, 2, 2, 2, 2, 8, 2, 2) & \mathbf{r}^1 &= (1, 2, \mathbf{5}, 3, 4, 5, 6, 7, 8, 1, 4, 4)\\
\mathbf{d}^2 &= (2, 3, 2, \mathbf{1}, 4, 2, 2, 2, 2, 2, 8, 2, 2) & \mathbf{r}^2 &= (1, 2, 5, \mathbf{8}, 3, 4, 5, 6, 7, 8, 1, 4, 4)\\
\mathbf{d}^3 &= (2, 3, 3, \mathbf{1}, 2, 4, 2, 2, 2, 2, 2, 8, 2, 2) & \mathbf{r}^3 &= (1, 2, 5, \mathbf{13}, 8, 3, 4, 5, 6, 7, 8, 1, 4, 4)\\
\mathbf{d}^4 &= (2, 3, 3, 1, 2, 5, \mathbf{1}, 3, 2, 2, 2, 2, 8, 2, 2) & \mathbf{r}^4 &= (1, 2, 5, 13, 8, 3, \mathbf{7}, 4, 5, 6, 7, 8, 1, 4, 4).
\end{aligned}$$

Therefore, $\mathbf{Sub}((\mathbf{d}^0, \mathbf{r}^0),\ \mathbf{b})$ is the following structure on $\mathrm{CT}(12, 3)$:

$$((2, 3, 3, 1, 2, 5, 1, 3, 2, 2, 2, 2, 8, 2, 2), (1, 2, 5, 13, 8, 3, 7, 4, 5, 6, 7, 8, 1, 4, 4)).$$

Next we define the ballot numbers, a generalization of the Catalan numbers, where the nth *Catalan number* (for $n \geq 1$) is defined by [7, A000108]:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The ballot numbers, which we denote by $B(n, k)$, count the number of lattice paths from $(0, 0)$ to (n, k) that do not cross above the line $y = x$. For more on ballot numbers see [4].

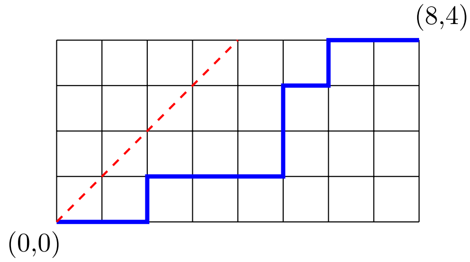


Figure 8: A lattice path starting at $(0,0)$ and ending at $(8,4)$, which is associated to the subdivision sequence $\mathbf{b} = (3,4,4,7)$.

Definition 9. For all $n, k \in \mathbb{N} \cup \{0\}$, with $0 \leq k \leq n$, define the *ballot numbers* [7, A009766] by

$$B(n,k) := \frac{n-k+1}{n+1}\binom{n+k}{n}.$$

Example 5. Consider the lattice path from $(0,0)$ to $(8,4)$ depicted in Figure 8. We now propose a subdivision sequence that is in bijection with these lattice paths. We want a vector with k parts (subdivisions) such that the value of every part satisfies $1 \leq b_i \leq p+i-1$.

We do so by the following process:

1. First, identify every "north" step we take and place the corresponding x-coordinate in a vector $(x_1, x_2, \ldots, x_k)$.

2. Then, subtract each value of the vector from $n+1$.

3. Place the resulting values in ascending order in a vector, $\mathbf{b} = (b_1, b_2, \ldots, b_k)$.

Let us return to Figure 8, and find its corresponding subdivision sequence. The 4 north steps come at the x-coordinates 2, 5, 5, and 6. So we get the vector $(2,5,5,6)$. Then we subtract each coordinate from 9 to get $9-2 = 7, 9-5 = 4, 9-5 = 4$ and $9-6 = 3$ and arrange them in ascending order to get $\mathbf{b} = (3,4,4,7)$. This is a valid subdivision sequence because each b_i is between 1 and $8+i-1$. Note that each lattice path takes exactly k north steps. Due to the line $y = x$, there can never be more than 1 north step at $x = 1$, 2 north steps at $x = 2$, and so on. The values also can never go above n. These are the exact conditions for a subdivision sequence with k subdivisions and maximum value n. Hence, the number of vectors $\mathbf{b}$ which are valid subdivision sequences with k subdivisions and maximum value n is the same as the number of lattice paths from $(0,0)$ to (n,k) that do not go above $y = x$.

The next result is used to count the number of arithmetical structures on $\mathrm{CT}(p,s)$ as we find a bijection between the subdivision sequences on smaller coconut trees and non-decreasing sequences, similar to the method used in Archer et al. [1].

Proposition 4. *Fix $s \geq 2$. Let $1 \leq i \leq p$ and let $(\mathbf{d},\mathbf{r})$ be any smooth arithmetical structure on $\mathrm{CT}(i,s)$. The number of arithmetical structures on $\mathrm{CT}(p,s)$ that are descendants of $(\mathbf{d},\mathbf{r})$ is $B(p-1,p-i)$.*

Proof. The proof follows in a similar manner to Proposition 2.11 in [1]. □

Next, we count the number of arithmetical structures on $\mathrm{CT}(p,s)$ with non-smoothable leaf nodes, that is, when $d_{\ell_j} > 1$ for $j \in \{1,2,\ldots,s\}$.

Corollary 1. *Fix $p \geq 1, s \geq 2$. The number of arithmetical structures on $\mathrm{CT}(p,s)$ such that all of the leaf vertices v_{ℓ_j} for $j \in \{1,2,\ldots,s\}$ cannot be smoothed is given by*

$$\sum_{i=1}^{p} B(p-1,p-i)|\boldsymbol{SArith(}\mathrm{CT}(i,s)\boldsymbol{)}|.$$

Proof. Since Proposition 4 shows that each smooth structure on $\mathrm{CT}(i,s)$ has $B(p-1,p-i)$ descendant arithmetical structures on $\mathrm{CT}(p,s)$, we iterate through every smooth structure on $\mathrm{CT}(i,s)$ from $i=1$ to $i=p$ and count the number of descendant arithmetical structures to get the total count of structures on $\mathrm{CT}(p,s)$ such that the leaf vertices cannot be smoothed. Thus, the number of arithmetical structures on $\mathrm{CT}(p,s)$ with $d_{\ell_1},\ldots,d_{\ell_k} \geq 2$ is given by

$$\sum_{i=1}^{p} B(p-1,p-i)|\mathbf{SArith(}\mathrm{CT}(i,s)\mathbf{)}|.$$ □

Example 6. Referring back to Figure 8, each lattice path corresponds to a unique descendant of a smooth arithmetical structure $(\mathbf{d},\mathbf{r})$ on $\mathrm{CT}(5,s)$. The descendant structures are on $\mathrm{CT}(9,s)$. There are

$$B(p-1,p-i) = B(9-1,9-5) = B(8,4) = 275$$

such descendants.

Proposition 5. *Let $p \geq 1$. The number of arithmetical structures on $\mathrm{CT}(p,1)$ such that the leaf vertex v_{ℓ_1} cannot be smoothed is given by*

$$|\boldsymbol{Arith(}\mathcal{P}_{p+1}\boldsymbol{)}| - |\boldsymbol{Arith(}\mathcal{P}_{p}\boldsymbol{)}| = C_p - C_{p-1},$$

where C_n is the nth Catalan number.

Proof. Braun et al. in [3, Lemma 1] proved that an arithmetical structure on a path must begin and end with 1. Thus, if v_{ℓ_1} can be smoothed, then $r_{\ell_1} = r_p = 1$. The arithmetical structures on $\mathrm{CT}(p,1) \cong \mathcal{P}_{p+1}$ with $r_{\ell_1} = r_p = 1$ are in bijection with arithmetical structures on $\mathcal{P}_p$ via smoothing at v_{ℓ_1}. Hence, there are $|\mathbf{Arith}(\mathcal{P}_p)| = C_{p-1}$ many of them. Therefore, there are $|\mathbf{Arith}(\mathcal{P}_{p+1})| - |\mathbf{Arith}(\mathcal{P}_p)| = C_p - C_{p-1}$ many arithmetical structures on $\mathrm{CT}(p,1)$ with the property that the leaf vertex cannot be smoothed. □

Note that for $s = 0$, there are no leaf vertices that can be smoothed. Thus, the number of arithmetical structures on $\mathrm{CT}(p,0) = \mathcal{P}_p$ is $|\mathbf{Arith}(\mathcal{P}_p)| = C_{p-1}$. Next, we prove the main result of the paper, Theorem 1, providing a count for the number of arithmetical structures on $\mathrm{CT}(p,s)$ in terms of smooth arithmetical structures as presented in Section 1.

Proof of Theorem 1. We begin by counting the number of arithmetical structures on $\mathrm{CT}(p,s)$ by enumerating arithmetical structures on $\mathrm{CT}(p,s)$ that have j leaf vertices (among the s vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_s}$) that can be smoothed for $0 \le j \le s$.

For $0 \le j \le s-2$, by Corollary 1, the number of arithmetical structures on $\mathrm{CT}(p, s-j)$ such that all $s-j$ leaf vertices cannot be smoothed is given by

$$\begin{aligned}
&\sum_{k=1}^{p} B(p-1,p-k)|\mathbf{SArith}(\mathrm{CT}(k,s-j))| \\
&\quad = \sum_{i=s-j+1}^{p+(s-j)} B(p-1,p-i+s-j)|\mathbf{SArith}(\mathrm{CT}(i-(s-j),s-j))| \\
&\quad = A(p+s-j,s-j).
\end{aligned}$$

Given one such structure $(\mathbf{d},\mathbf{r})$ on $\mathrm{CT}(p,s-j)$, we can construct $\binom{s}{j}$ structures in $\mathrm{CT}(p,s)$ by choosing j leaf vertices and setting their r-value to r_p, placing the labels $r_{\ell_1}, r_{\ell_2}, \ldots, r_{\ell_{s-j}}$ (none of which equals r_p) in the remaining $s-j$ leaves in the order listed, and keeping the values of $r_1, r_2, \ldots, r_p$ intact in the first p vertices. Thus, the total number of arithmetical structures with j vertices that can be smoothed is given by $\binom{s}{j} A(p+s-j, s-j)$. Taking the sum over the possible values of j gives all but the last two terms on the right-hand side of Equation (1). We consider those values for j next.

If $j = s-1$, then $\mathrm{CT}(p,s-j) = \mathrm{CT}(p,1)$. By Proposition 5, the number of arithmetical structures on $\mathrm{CT}(p,1)$ such that the leaf vertex v_{ℓ_1} cannot be smoothed is $C_p - C_{p-1} = A(p+1,1)$. Similarly to the previous case, given one such structure $(\mathbf{d},\mathbf{r})$ in $\mathrm{CT}(p,1)$, we can construct $\binom{s}{s-1}$ structures in $\mathrm{CT}(p,s)$ by choosing $s-1$ leaf vertices and setting their r-value to r_p, placing the label r_{ℓ_1} (that is not equal to r_p) in the remaining leaf vertex, and keeping the values of $r_1, r_2, \ldots, r_p$ intact in the first p vertices.

If $j = s$, then $\mathrm{CT}(p, s-j) = \mathrm{CT}(p, 0)$. Hence, the number of arithmetical structures on $\mathrm{CT}(p,0)$ with no leaf vertices that can be smoothed is simply the number of structures on $\mathrm{CT}(p,0) = \mathcal{P}_p$, which is given by $C_{p-1} = A(p,0)$. Similar to previous cases, given one such structure, we can place the label r_p on every one of the s leaf vertices of $\mathrm{CT}(p,s)$, to get a structure on $\mathrm{CT}(p,s)$. This gives the final correspondence to get that

$$|\mathbf{Arith}(\mathrm{CT}(p,s))| = \sum_{j=0}^{s} \binom{s}{j} A(p+s-j, s-j). \qquad \square$$

We now confirm the count for the number of arithmetical structures on a coconut tree $\mathrm{CT}(p,2)$ as given in [1, Theorem 2.12].

Corollary 2. *If $p \geq 2$, then*

$$|\boldsymbol{Arith}(\mathrm{CT}(p,2))| = 2C_p - C_{p-1} + \sum_{i=3}^{p+2} B(p-1, p+2-i)|\boldsymbol{SArith}(\mathrm{CT}(i-2,2))|.$$

Proof. By Theorem 1 we have

$$\begin{aligned}
|\mathbf{Arith}(\mathrm{CT}(p,2))| &= \sum_{j=0}^{2} \binom{2}{j} A(p+2-j, 2-j),\\
&= \binom{2}{0} A(p+2,2) + \binom{2}{1} A(p+1,1) + \binom{2}{2} A(p,0)\\
&= \left(\sum_{i=3}^{p+2} B(p-1,p+2-i)|\mathbf{SArith}(\mathrm{CT}(i-2,2))|\right) + 2(C_p - C_{p-1}) + C_{p-1}\\
&= 2C_p - C_{p-1} + \sum_{i=3}^{p+2} B(p-1,p+2-i)\cdot|\mathbf{SArith}(\mathrm{CT}(i-2,2))|. \qquad \square
\end{aligned}$$

The following is a derivation of the number of arithmetical structures on a star graph. Let $\mathcal{S}_s$ denote the star graph on $s+1$ vertices, which is equivalent to $\mathrm{CT}(1,s)$. The next result shows that the number of arithmetical structures on a star graph can be computed using the number of smooth arithmetical structures on smaller star graphs.

Corollary 3. *If $p = 1$ and $s \geq 2$, then*

$$|\boldsymbol{Arith}(\mathrm{CT}(1,s))| = 1 + \sum_{j=0}^{s-2} \binom{s}{j} |\boldsymbol{SArith}(\mathcal{S}_{s-j})|.$$

Proof. Using Theorem 1, we have

$$
\begin{aligned}
&|\mathbf{Arith}(\mathrm{CT}(1,s))| \\
&= \sum_{j=0}^{s} \binom{s}{j} A(s+1-j, s-j) \\
&= \binom{s}{s} A(1,0) + \binom{s}{s-1} A(2,1) + \sum_{j=0}^{s-2} \binom{s}{j} A(s+1-j, s-j) \\
&= 1 \cdot 1 + s \cdot 0 \\
&\quad + \sum_{j=0}^{s-2} \binom{s}{j} \left(\sum_{i=s+1-j}^{s+1-j} B((i-(s-j)-1, (s+1-j)-i)|\mathbf{SArith}(\mathrm{CT}(i-(s-j), s-j))| \right) \\
&= 1 + \sum_{j=0}^{j=s-2} \binom{s}{j} B(0,0)|\mathbf{SArith}(\mathrm{CT}(1, s-j))| \\
&= 1 + \sum_{j=0}^{j=s-2} \binom{s}{j} |\mathbf{SArith}(\mathrm{CT}(1, s-j))|. \qquad \square
\end{aligned}
$$

4.1. Counting Smooth Arithmetical Structures

In this subsection, we give an explicit construction of smooth arithmetical structures on coconut tree graphs and give some enumeration results. First, we define Euclidean chains, which are sequences that capture the construction of arithmetical structures on paths. As an important notation convention, in this section, when we write $c = a \mod b$ we take c to be the smallest nonnegative representative in the class of $a \mod b$. In particular, c is always less than b.

Definition 10. A *Euclidean chain* is a sequence $\{x_i\}_{i\in\mathbb{N}}$ defined as follows: $x_1, x_2 \in \mathbb{N}$ and for all $i \geq 2$,

$$
x_{i+1} = \begin{cases} -x_{i-1} \mod x_i & \text{if } x_i \neq 0, 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Then, the *Euclidean chain function* $F : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \to \mathbb{N}$ is defined as the function $F(x_1, x_2) = k$ where k is the largest value of i such that x_i is nonzero, or the number of positive terms in the sequence $\{x_i\}$. Since $x_{i+1} = -x_{i-1} \mod x_i < x_i$, then the sequence eventually terminates and F is well defined.

Example 7. We calculate $F(13, 60)$. Take $x_1 = 13$ and $x_2 = 60$. Let x_3 be the least residue of $-13 \pmod{60} \equiv 47 \pmod{60}$, which gives us $47 = x_3$. Likewise, let us calculate the remaining x_i for $i = 3, \ldots, 9$, which are found as follows:

$$\begin{array}{llllll} -13 & \pmod{60} & \equiv & 47 & \pmod{60} & \Rightarrow & x_3 = 47 \\ -60 & \pmod{47} & \equiv & 34 & \pmod{47} & \Rightarrow & x_4 = 34 \\ -47 & \pmod{34} & \equiv & 21 & \pmod{34} & \Rightarrow & x_5 = 21 \\ -34 & \pmod{21} & \equiv & 8 & \pmod{21} & \Rightarrow & x_6 = 8 \\ -21 & \pmod{8} & \equiv & 3 & \pmod{8} & \Rightarrow & x_7 = 3 \\ -8 & \pmod{3} & \equiv & 1 & \pmod{3} & \Rightarrow & x_8 = 1 \\ -3 & \pmod{1} & \equiv & 0 & \pmod{1} & \Rightarrow & x_9 = 0. \end{array}$$

Hence, $F(13, 60) = 8$ as x_8 is the last x_i with a non-zero value.

Next, we give the following proposition from Archer et al. [1] that greatly simplifies calculations involving the Euclidean chain function F.

Proposition 6 ([1, Lemma 3.2])**.** *Let $x \in \mathbb{N}$ and $y, k \in \mathbb{N} \cup \{0\}$. Then,*

1. $F(x + ky, y) = F(x, y)$
2. $F(x, kx + y) = F(x, y) + k$.

Corollary 4. *Let $x, y \in \mathbb{N}$ such that $x \mid y$. Then, $F(x, y) = 1 + \frac{y}{x}$.*

Proof. Since $x \mid y$, there exists some $n \in \mathbb{N}$ such that $nx = y$. Then, note that $F(x, y) = F(x, nx + 0) = F(x, 0) + n$, where we used property (2) of Proposition 6. Since $nx = y$ yields $n = \frac{y}{x}$ and by definition $F(x, 0) = 1$, we have $F(x, y) = 1 + \frac{y}{x}$. □

Corollary 5. *If $y \in \mathbb{N} \cup \{0\}$, then $F(1, y) = y + 1$ and if $x \in \mathbb{N}$, then $F(x, 1) = 2$.*

Proof. Since $1 \mid y$, we use Corollary 4 to arrive $F(1, y) = 1 + \frac{y}{1} = y + 1$. For the second statement, if a Euclidean chain starts with $\{x, 1\}$ then, by definition, the next term is 0 and the sequence is $\{x, 1, 0, 0, \ldots\}$. Thus, $F(x, 1) = 2$. □

In what follows, we begin by assuming that we have assigned labels to the leaf vertices such that they divide the assigned label on the center vertex of a coconut tree graph. Our main result establishes that given this initial labeling, we can construct a unique smooth arithmetical structure on a coconut tree whose path length is one less than the Euclidean chain evaluated at the sum of the leaf labels and the label of the center vertex (Corollary 6). Before we establish this result, we illustrate the procedure.

Example 8. Let $(c, a_1, a_2, a_3, a_4) = (60, 2, 3, 3, 5)$. Note that $\sum_{j=1}^{4} a_j = 13$ and by Example 7, we have $F(13, 60) = 8$. Hence, we will construct a smooth arithmetical structure on $\mathrm{CT}(8 - 1, 4)$ with $r_7 = 60$ and $r_{\ell_j} = a_j$ for $j = 1, 2, 3, 4$. To determine $r_1, r_2, \ldots, r_6$, we use the positive entries of the Euclidean chain that starts $\{13, 60\}$ in reverse order. By Example 7, this chain is $\{13, 60, 47, 34, 21, 8, 3, 1, 0, 0, \ldots\}$. Hence, we let

$$\mathbf{r} = (1, 3, 8, 21, 34, 47, 60, 2, 3, 3, 5),$$

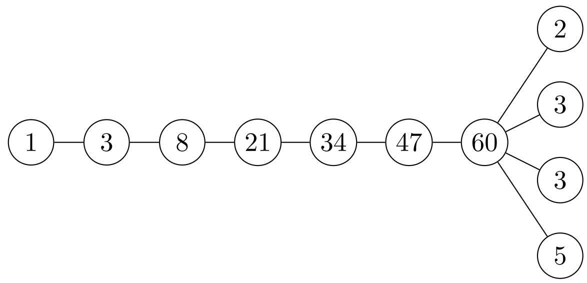


Figure 9: The constructed Coconut Tree $\mathrm{CT}(7,4)$.

which is illustrated in Figure 9.

Proposition 7. *For every tuple* $(c, a_1, a_2, \ldots, a_s) \in \mathbb{N}^{s+1}$ *such that* $a_i \mid c$, $a_i < c$ *for* $i \in \{1, 2, \ldots, s\}$, *and* $\gcd(c, a_1, a_2, \ldots, a_s) = 1$, *there is a unique* $p \geq 1$ *such that there is a smooth arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on* $\mathrm{CT}(p, s)$ *with* $r_{\ell_i} = a_i$ *for* $i \in \{1, 2, \ldots, s\}$ *and* $r_p = c$. *Moreover, the arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *is also unique.*

Proof. Let $(c, a_1, a_2, \ldots, a_s) \in \mathbb{N}^{s+1}$ such that $\gcd(c, a_1, a_2, \ldots, a_k) = 1$, $a_i \mid c$ and $c > a_i$ for $i \in \{1, 2, \ldots, s\}$. Let $r_{\ell_i} = a_i$ for $i \in \{1, 2, \ldots, s\}$. Define the sequence $\{x_i\}_{i=1}^{\infty}$ as follows:

$$x_i = \begin{cases} c & \text{if } i = 1, \\ \left(-\sum_{j=1}^{s} a_j\right) \mod c & \text{if } i = 2, \\ -x_{i-2} \mod x_{i-1} & \text{if } i \geq 3 \text{ and } x_{i-1}, x_{i-2} \neq 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

Hence, $\{x_i\}_{i=1}^{\infty}$ forms a decreasing sequence and moreover, it is a Euclidean chain. Let $p = F\left(c, \left(-\sum_{j=1}^{s} a_j\right) \mod c\right)$, that is, let p be the number of nonzero entries of $\{x_i\}_{i=1}^{\infty}$. Let $r_{p+1-i} = x_i$ for $i \in \{1, 2, \ldots, p\}$ such that $x_i \neq 0$. Then, we verify the resulting $\mathbf{r}$-vector, namely $\mathbf{r} = (r_1, r_2, \ldots, r_p, a_1, a_2, \ldots, a_s)$, is an arithmetical structure. By assumption, $r_{\ell_i} = a_i$ and since $a_i \mid c = r_p$, then $r_{\ell_i} \mid r_p$. For v_p, note that $r_{p-1} = \left(-\sum_{j=1}^{s} a_j\right) \mod r_p$, hence $r_p \left| \left(\sum_{j=1}^{s} r_{\ell_j}\right) + r_{p-1}\right.$. For the vertices $v_2, \ldots, v_{p-1}$, we constructed the $\mathbf{r}$ labels such that $r_{i+1} = -r_{i-1} \mod r_i$ hence $r_i \mid (r_{i-1} + r_{i+1})$. Finally, for v_1, since $r_1 = x_p$, it is the last nonzero entry of the Euclidean chain $\{x_i\}_{i=1}^{\infty}$. Thus, $x_{p+1} = 0$, which means that either $x_p = 1$ or $-x_{p-1} \mod x_p = 0$. In both cases, we get that $x_p \mid x_{p-1}$, that is, $r_1 \mid r_2$. Hence, the constructed $\mathbf{r}$-vector determines an arithmetical structure on $\mathrm{CT}(p, s)$.

Now, we show this arithmetical structure on $\mathrm{CT}(p, s)$ is smooth. Note that by assumption $r_{\ell_i} < r_p$ and $\gcd(r_p, r_{\ell_1}, r_{\ell_2}, \ldots, r_{\ell_s}) = 1$. Since the entries of the tuple $(r_p, r_{p-1}, \ldots, r_1)$ are consecutive entries of the Euclidean chain $\{x_i\}_{i=1}^{\infty}$, then $r_p > r_{p-1} > \cdots > r_1$. By Lemma 2, we get that $\mathbf{r}$ is a smooth arithmetical structure. The uniqueness of p (and of $\mathbf{r}$) comes from the fact that in order to have $r_p > r_{p-1} > \cdots > r_1$ and $r_i \mid (r_{i-1} + r_{i+1})$ for $i = 1, 2, \ldots, p$ (denoting $r_0 = 0$ and $r_{p+1} = \sum_{j=1}^{s} r_{\ell_j}$), we must have $r_i = -r_{i+2} \mod r_{i+1}$, hence the values of $\mathbf{r}$ must be the entries of the Euclidean chain $\{x_i\}_{i=1}^{\infty}$. □

The next corollary reduces the problem of bounding the number of smooth arithmetical structures that can be generated from only the leaf vertices to studying the Euclidean chain function F.

Corollary 6. *Let* $(c, a_1, a_2, \ldots, a_s) \in \mathbb{N}^{s+1}$ *such that* $\gcd(c, a_1, a_2, \ldots, a_s) = 1$, $a_i \mid c$*, and* $a_i < c$ *for all* $i \in \{1, 2, \ldots, s\}$*. The smooth arithmetical structure on the coconut tree* $\mathrm{CT}(p, s)$ *constructed from Proposition 7 satisfies*

$$p = F\left(\left(\sum_{j=1}^{s} a_j\right), c\right) - 1.$$

Proof. Note that by the construction of the Euclidean chain $\{x_i\}$ in Proposition 7, the number of vertices that are constructed is

$$F\left(c, -\left(\sum_{j=1}^{s} a_j\right) \mod c\right) = F\left(\left(\sum_{j=1}^{s} a_j\right), c\right) - 1. \qquad \square$$

Next, we show that having only assigned labels to the leaf vertices is not enough to guarantee a unique smooth arithmetical structure.

Example 9. Let $c = 6$, and let $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 3, 3, 3)$. Using Corollary 6 followed by Proposition 6 we have $p = F(13, 6) - 1 = F(1, 6) - 1$. Then by Corollary 5, $F(1, 6) = 7$, so $p = 6$. On the other hand, if $c = 12$, then we have $(c, a_1, a_2, a_3, a_4, a_5) = (12, 2, 2, 3, 3, 3)$. Using the same results, we have $p = F(13, 12) - 1$, which then utilizing the fact that $F(13, 12) = F(1, 12)$, results in $p = 12$. Moreover, the resulting arithmetical structures on $\mathrm{CT}(6, 5)$ and $\mathrm{CT}(12, 5)$, have the path labels 1 through 6, and 1 through 12, respectively, listed in increasing order.

Proposition 8. *Let* $p \geq 1$*. Any smooth arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on* $\mathrm{CT}(p, s)$ *has* $d_p > 1$ *if and only if* $r_p < \sum_{j=1}^{s} r_{\ell_j}$.

Proof. Let $(\mathbf{d}, \mathbf{r})$ be a smooth arithmetical structure on $\mathrm{CT}(p, s)$ with $d_p > 1$. If $p = 1$, then

$$\sum_{j=1}^{s} r_{\ell_j} = d_p r_p > r_p.$$

If $p > 1$, then

$$d_p r_p = r_{p-1} + \sum_{j=1}^{s} r_{\ell_j}.$$

Since the arithmetical structure is smooth, by Lemma 1, we have $r_p > r_{p-1}$. Thus,

$$d_p r_p = r_{p-1} + \sum_{j=1}^{s} r_{\ell_j} < r_p + \sum_{j=1}^{s} r_{\ell_j}.$$

Subtracting r_p from both sides and combining terms yields

$$\sum_{j=1}^{s} r_{\ell_j} > (d_p - 1) r_p > r_p.$$

For the reverse direction, assume that $r_p < \sum_{j=1}^{s} r_{\ell_j}$. Note that $p > 1$, because $r_p \neq \sum_{j=1}^{s} r_{\ell_j}$. Then

$$d_p r_p = r_{p-1} + \sum_{j=1}^{s} r_{\ell_j} > r_{p-1} + r_p > r_p,$$

which implies that $d_p > 1$. □

Next, we give an alternative proof to Proposition 2.3 in Archer et al. [1], which stated that any smooth arithmetical structures on bidents, denoted as $\mathrm{CT}(p, 2)$, must have center vertex equal to 1.

Corollary 7. *Let $p \geq 1$. There are no smooth arithmetical structures $(\mathbf{d}, \mathbf{r})$ on $\mathrm{CT}(p, 2)$ with $d_p > 1$.*

Proof. Let $(\mathbf{d}, \mathbf{r})$ be a smooth arithmetical structure on $\mathrm{CT}(p, 2)$. Then r_{ℓ_1} and r_{ℓ_2} are proper divisors of r_p. By Proposition 8, $d_p > 1$ if and only if $r_{\ell_1} + r_{\ell_2} > r_p$. So we need $r_{\ell_1} + r_{\ell_2} > r_p$. However, since r_{ℓ_1} and r_{ℓ_2} are proper divisors of r_p, we have $r_{\ell_1} + r_{\ell_2} \leq r_p$. Therefore, d_p can not be greater than 1. □

Next, given the numbers assigned to leaf vertices, we count the number of smooth arithmetical structures that have a label greater than one at the central vertex.

Proposition 9. *Let $(r_{\ell_1}, \ldots, r_{\ell_s}) \in \mathbb{N}^s$, let $S = \{n \in \mathbb{N} \mid n \cdot \mathrm{lcm}(r_{\ell_1}, \ldots, r_{\ell_s}) < \sum_{j=1}^{s} r_{\ell_j}\}$. Then, given $(r_{\ell_1}, \ldots, r_{\ell_s})$, the number of smooth arithmetical structures that can be constructed such that the central vertex has label greater than 1 is given by*

$$|S| = \left\lfloor \frac{\sum_{j=1}^{s} r_{\ell_j}}{\mathrm{lcm}(r_{\ell_1}, \ldots, r_{\ell_s})} \right\rfloor.$$

Proof. First, consider when $|S| = 0$, which implies $\text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s}) \geq \sum_{j=1}^{s} r_{\ell_j}$. If there is a structure $(\mathbf{d}, \mathbf{r})$ on $\text{CT}(p, s)$ for some $p \in \mathbb{N}$, then since each r_{ℓ_j} divides r_p, we have $\text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s}) \mid r_p$. Thus, $r_p \geq \text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s}) \geq \sum_{j=1}^{s} r_{\ell_j}$. This contradicts Proposition 8, hence there exists no smooth arithmetical structures that can be constructed such that $d_p > 1$.

Now, consider when $|S| \geq 1$. Let $r_{p_i} = i \cdot \text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s})$ for $i = 1, 2, \ldots, |S|$. By Proposition 7, we can construct $|S|$ smooth arithmetical structures, one per each value of r_{p_i} assigned to the central vertex. By Proposition 8, each of these structures have $d_{p_i} > 1$. Moreover, since in any structure of the form

$$(r_1, r_2, \ldots, r_p, r_{\ell_1}, r_{\ell_2}, \ldots, r_{\ell_s})$$

we must have $r_p \mid \text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s})$, the only values of r_p that satisfy that $r_p < \sum_{j=1}^{s} r_{\ell_j}$ are r_{p_i} for $i = 1, 2, \ldots, |S|$. Then by Proposition 8 these are the only possible smooth structures with $d_p > 1$. To complete the proof, note that by the definition of S, $|S|$ is the largest integer m such that $m < \frac{\sum_{j=1}^{s} r_{\ell_j}}{\text{lcm}(r_{\ell_1}, r_{\ell_s})}$. Hence, $|S| = \left\lfloor \frac{\sum_{j=1}^{s} r_{\ell_j}}{\text{lcm}(r_{\ell_1}, \ldots, r_{\ell_s})} \right\rfloor$. □

We conclude with an application of Proposition 9.

Example 10. Recalling Example 9, note that $\text{lcm}(r_{\ell_1}, \ldots, r_{\ell_5}) = 6$ and $\sum_{j=1}^{5} r_{\ell_j} = 13$. Thus, the number of smooth arithmetical structures with label at the center vertex greater than 1 is $\lfloor \frac{13}{6} \rfloor = 2$, which corresponds to when $r_{p_1} = 6$ and $r_{p_2} = 12$. These are precisely the two structures presented in Example 9, namely,

$$\mathbf{r} = (1, 2, 3, 4, 5, 6, 2, 2, 3, 3, 3) \text{ and } \mathbf{r} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 2, 2, 3, 3, 3).$$

5. Future Work

We conclude with some direction for future research. We recall that Archer et al. [1] establish that the number of smooth arithmetical structures on a bident is bounded by a cubic polynomial. We ask the following questions.

Question 1. Does there exist a polynomial bound for the number of smooth arithmetical structures on coconut trees $\text{CT}(p, s)$ for $s > 2$?

Question 2. If a polynomial bound for the number of smooth arithmetical structures on coconut trees $\text{CT}(p, s)$ for $s > 2$ exists, how does it relate to s the number of leaves?

We now pose a number theoretic question which would help in enumerating smooth arithmetical structures on $\text{CT}(p, s)$ if we are given r_p and the sum of the leaves.

Question 3. Given integers r_p and $\sum_{j=1}^{s} r_{\ell_j}$, in how many ways can we partition $\sum_{j=1}^{s} r_{\ell_j}$ using the proper divisors of r_p?

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