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EXPLICIT EVALUATION OF A QUADRATIC SUM OVER CONGRUENT PAIRS MODULO A PRIME

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Abstract

Let p be an odd prime number and let l be a positive integer such that $\gcd(p, l) = 1$. The main purpose of this note is to prove an identity for the sum

$$\sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ b \equiv al \pmod p}}^{p-1} \left(\frac{a}{p} - \frac{1}{2}\right)\left(\frac{b}{p} - \frac{1}{2}\right).$$

1. Introduction and Main Result

Let p be an odd prime number and let l be a positive integer such that $\gcd(p, l) = 1$. Set

$$F(p, l) := \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ b \equiv al \pmod p}}^{p-1} \left(\frac{a}{p} - \frac{1}{2}\right)\left(\frac{b}{p} - \frac{1}{2}\right).$$

In particular, if $l = 1$, then we obtain the following simple closed-form expression:

$$F(p, 1) = \sum_{a=1}^{p-1} \left(\frac{a}{p} - \frac{1}{2}\right)^2 = \frac{(p-1)(p-2)}{12p},$$

which naturally motivates the question of whether similar closed forms exist for general l coprime to p.

The aim of this short note is to establish such a closed formula for $F(p, l)$ in terms of p, l, and a trigonometric sum involving cotangent functions, through the following theorem.

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Theorem 1. *Let $p \geq 3$ be a prime and l be a positive integer such that $\gcd(p,l) = 1$. Then*

$$F(p,l) = \frac{p^2 - 3lp + l^2 + 1}{12lp} - s(p,l), \tag{1}$$

where $s(p,l)$, which depends only on the value of $p \mod l$, is the Dedekind sum

$$s(p,l) = \frac{1}{4l} \sum_{a=1}^{l-1} \cot\left(\frac{\pi a}{l}\right) \cot\left(\frac{\pi a p}{l}\right).$$

Example 2. Let $p \geq 3$ be a prime. Then

$$F(p,p-1) = \frac{(p-1)(2-p)}{12p},$$
$$F(p,p+1) = \frac{(p-1)(p-2)}{12p}.$$

Example 3. Let $p \geq 3$ be a prime. Then

$$F(p,2) = \frac{(p-1)(p-5)}{24p};$$

in particular, $F(5,2) = 0$.

Example 4. Let $p \geq 3$ be a prime. Then

$$F(p,3) = \frac{1}{36p} \times \begin{cases} (p-1)(p-10), & \text{if } p \equiv 1 \pmod 3 \\ (p-2)(p-5), & \text{if } p \equiv 2 \pmod 3 \end{cases};$$

in particular, $F(5,3) = 0$.

Example 5. Let $p \geq 3$ be a prime. Then

$$F(p,4) = \frac{1}{48p} \times \begin{cases} (p-1)(p-17), & \text{if } p \equiv 1 \pmod 4 \\ p^2 - 6p + 17, & \text{if } p \equiv 3 \pmod 4 \end{cases};$$

in particular, $F(17,4) = 0$.

Example 6. Let $p \geq 3$ be a prime. According to [3, Proposition 1] we have

$$F(p,8) = \frac{1}{96p} \times \begin{cases} p^2 - 66p + 5, & \text{if } p \equiv 1 \pmod 8 \\ p^2 - 30p + 65, & \text{if } p \equiv 3 \pmod 8 \\ (p-5)(p-13), & \text{if } p \equiv 5 \pmod 8 \\ (p+5)(p+13), & \text{if } p \equiv 7 \pmod 8 \end{cases};$$

in particular, $F(5,8) = F(13,8) = 0$.

Moreover, one can use the results of Louboutin in [3] to derive closed-form expressions for $F(p,l)$ with $l \in \{5, 6, 10\}$.

2. Proof of Theorem 1.

We now present the proof of Theorem 1.

Proof of Theorem 1. Let l be a positive integer and let $p \geq 3$ be an odd prime that does not divide l. Set

$$M(p,l) := \frac{2}{p-1} \sum_{\chi \in G_p^-} \chi(l) L(1,\chi) L(1,\overline{\chi}),$$

where G_p^- denotes the set of cardinality $(p-1)/2$ of the odd Dirichlet characters modulo p. On the one hand, according to [2, Theorem 1] we have

$$M(p,l) = \frac{\pi^2(p^2 - 3(l + 4l\ s(p,l))p + l^2 + 1)}{6lp^2}. \tag{2}$$

On the other hand, for any odd primitive character χ, it is known (see, e.g., [4, Theorem 4.9]) that

$$L(1,\chi) = \frac{\pi i}{p} \tau(\chi)\, B_1(\overline{\chi}),$$

where $\tau(\chi)$ and $B_1(\chi)$ denote the Gaussian sum and the generalized Bernoulli number associated with χ, respectively. Substituting this into the definition of $M(p,l)$, we obtain

$$M(p,l) = \frac{-2\pi^2}{p^2(p-1)} \sum_{\chi \in G_p^-} \chi(l)\tau(\chi)\tau(\overline{\chi}) B_1(\chi) B_1(\overline{\chi}). \tag{3}$$

Moreover, since

$$\tau(\chi)\tau(\overline{\chi}) = -p$$

(see, e.g., [4, Lemmas 4.7 and 4.8]), and since

$$B_1(\chi) = \sum_{a=1}^{p-1} \chi(a) \left(\frac{a}{p} - \frac{1}{2} \right)$$

(see, e.g., [4, Proposition 4.1]), Identity (3) becomes

$$M(p,l) = \frac{2\pi^2}{p(p-1)} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a}{p} - \frac{1}{2} \right) \left(\frac{b}{p} - \frac{1}{2} \right) \sum_{\chi \in G_p^-} \chi(al)\overline{\chi}(b). \tag{4}$$

Next, the orthogonality relations (see, e.g., [1, Theorem 6.16]) give us:

$$\sum_{\chi \in G_p^-} \chi(m)\overline{\chi}(n) = \begin{cases} (p-1)/2 & \text{if } n \equiv m \pmod p \text{ and } \gcd(m,p) = 1, \\ (1-p)/2 & \text{if } n \equiv -m \pmod p \text{ and } \gcd(m,p) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This fact allows us to write (4) as:

$$M(p,l) = \frac{\pi^2}{p}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ b\equiv al \pmod p}}^{p-1}\left(\left(\frac{a}{p}-\frac{1}{2}\right)-\left(\frac{p-a}{p}-\frac{1}{2}\right)\right)\left(\frac{b}{p}-\frac{1}{2}\right)$$

$$= \frac{2\pi^2}{p}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ b\equiv al \pmod p}}^{p-1}\left(\frac{a}{p}-\frac{1}{2}\right)\left(\frac{b}{p}-\frac{1}{2}\right). \tag{5}$$

Identities (2) and (5) together yield the desired result. This completes the proof of the theorem. □

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