ON NEAR F_k-PERFECT AND DEFICIENT F_k-PERFECT NUMBERS

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Abstract

For a positive integer n, the arithmetic function $\sigma_2(n)$ denotes the sum of squares of all the divisors of n. A positive integer n is called an F-perfect number if $\sigma_2(n) - n^2 = 3n$. A positive integer n is termed a near F-perfect number if $\sigma_2(n) - n^2 - d^2 = 3n$, where d is a proper divisor of n. Similarly, n is considered a deficient F-perfect number if $\sigma_2(n) - n^2 + d^2 = 3n$, where d is a proper divisor of n. In this paper, we discuss several characterizations of these numbers, establish their relations with other significant numbers, and generalize the near-perfect and deficient-perfect numbers.

1. Introduction

For a positive integer n, let $\sigma(n)$ denote the sum of its positive divisors. Positive integers can be categorized into three classes based on this divisor sum function. If

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$\sigma(n) = 2n$, then n is called a *perfect number*, and if $\sigma(n) > 2n$ (respectively, $\sigma(n) < 2n$), then n is called an *abundant* (respectively, *deficient*) number. For primes p and $2^p - 1$, an integer n is perfect if and only if $n = 2^{p-1}(2^p - 1)$. To the current day, no proof has been found to justify the existence of an odd perfect number. If an odd perfect number exists, it must be greater than 10^{1500} [2]. Pollock and Shevelev [3] introduced *near-perfect numbers*, defining them as positive integers n for which $\sigma(n) - d = 2n$, where d is a proper divisor of n known as the *redundant divisor* of n. In [3], they provided the construction of all near-perfect numbers with two prime factors. Min Tang et al. [4] proved that there is no odd near-perfect number with three prime factors and determined all deficient-perfect numbers with at most two distinct prime factors. They also proved that there is no odd deficient-perfect number with three prime factors. Xiaoyan Ma and Min Feng [5] demonstrated that the only odd near-perfect number with four distinct prime factors is $3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$. Similarly, Shichun Yang et al. [6] proved that the only odd deficient-perfect number with four prime factors is $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$.

For a positive integer n, the arithmetic function $\sigma_2(n)$ is defined as

$$\sigma_2(n) = \sum_{d|n} d^2.$$

Tianxin Cai et al. [1] introduced the concept of F-perfect numbers. A positive integer n is considered an *F-perfect number* if $\sigma_2(n) - n^2 = 3n$. Following the concept of abundant and deficient numbers, we call a positive integer n an *F-abundant* (respectively, *F-deficient*) number if $\sigma_2(n) - n^2 > 3n$ (respectively, $\sigma_2(n) - n^2 < 3n$). We have the following natural definitions.

Definition 1. A positive integer n is called a *deficient F-perfect* number if there exists a proper divisor d of n such that

$$\sigma_2(n) - n^2 + d^2 = 3n.$$

Definition 2. For any natural number $k > 1$, a positive integer n is called a *deficient F_k-perfect* number if there exists a proper divisor d of n such that

$$\sigma_k(n) - n^k + d^k = 3n.$$

It is easily observed that a deficient F_2-perfect number is nothing but a deficient F-perfect number. If n is a deficient F_k-perfect number, where $k > 1$, then n must satisfy the condition

$$\frac{\sigma_k(n)}{n} - n < 3.$$

Further generalizing, we define the following.

Definition 3. For any natural numbers k and ℓ, a positive integer n is called a *$[k, \ell]$-deficient-perfect* number if there exists a proper divisor d of n such that

$$\sigma_k(n) - n^k + d^k = \ell n.$$

Clearly, a deficient-perfect number is a $[1,1]$-deficient-perfect number.

Definition 4. A positive integer n is called a near *F-perfect* number if there exists a proper divisor d of n such that

$$\sigma_2(n) - n^2 - d^2 = 3n.$$

Generalizing near F-perfect numbers, we define the following.

Definition 5. For any natural number $k > 1$, a positive integer n is called a *near F_k-perfect* number if there exists a proper divisor d of n such that

$$\sigma_k(n) - n^k - d^k = 3n.$$

It is easily observed that a near F_2-perfect number is a near F-perfect number. If a positive integer n is a near F_k-perfect number, where $k \geq 1$ is a natural number, then n must satisfy the condition

$$\frac{\sigma_k(n)}{n} - n > 3.$$

Our final definition is the following.

Definition 6. For any natural number k and ℓ, a positive integer n is called a *$[k,\ell]$-near-perfect* number if there exists a proper divisor d of n such that

$$\sigma_k(n) - n^k - d^k = \ell n.$$

Clearly, a near-perfect number is a $[1,1]$-near-perfect number and a near F_k-perfect number is a $[k,3]$-near-perfect number for all $k \geq 2$. Finally, a near F-perfect number is a $[2,3]$-near-perfect number. Similarly, different kinds of near-perfect numbers can be extracted from $[k,\ell]$-near-perfect numbers.

Proposition 1. *An odd prime number is not a $[k,\ell]$-deficient-perfect number.*

Proof. Let an odd prime p be a $[k,\ell]$-deficient-perfect number. Then

$$\sigma_k(p) - p^k + d^k = \ell p.$$

Therefore,

$$1 + d^k = \ell p.$$

If $d = 1$, then $\ell = 1$ and $p = 2$, this is a contradiction as p is an odd prime. If $d = p$, then $1 + p^k = \ell p$, but $p \nmid 1 + p^k$. Therefore, the result follows. □

As a consequence of Proposition 1, we have the following corollaries.

Corollary 1. *The only even prime number 2 is a $[k,\ell]$-deficient-perfect number for $\ell = 1$.*

Corollary 2. *No prime number is deficient F-perfect.*

Proposition 2. *A prime number p is not a $[k,\ell]$-near-perfect number, for all natural numbers k and l.*

Proof. If a prime p is a $[k,\ell]$-near-perfect number, then

$$\sigma_k(p) - p^k - d^k = \ell p.$$

This implies,

$$1 - d^k = \ell p.$$

Therefore,

$$d^k = 1 - \ell p < 0, \ (p \geq 2, \ell \in \mathbb{N}).$$

This is a contradiction, hence the result follows. □

In this paper, we categorize the different types of positive integers defined above into two classes based on their prime factors. In Section 2, we characterize positive integers with a single prime factor. In Section 3, positive integers with two or more primes are characterized.

2. Characterization of Positive Integers that Factor into Single Prime Powers

In this section, to extend our understanding of the numbers introduced in Section 1, our goal is to characterize and deduce the properties of these numbers that have a single prime factor.

Theorem 1. *For any non-negative integer a, there is neither a deficient F-perfect number nor a near F-perfect number of the form $n = p^a$ for any prime p.*

Proof. Since the case for a near F-perfect number is similar, we only prove the result for deficient F-perfect number.

If $a = 0$, then $n = 1$ is trivially not a deficient F-perfect number. For $a = 1$, we see that $n = p$ is also not a deficient F-perfect number by Corollary 2. For any prime p, let $n = p^a$ be a deficient F-perfect number where $a \geq 2$. Then, we have

$$\sigma_2(p^a) - p^{2a} + p^{2b} = 3p^a, \text{ where } b < a.$$

Therefore,

$$1 + p^2 + p^4 + \cdots + p^{2(a-1)} + p^{2b} = 3p^a. \tag{1}$$

We may consider two cases.

`Case 1:` $b \geq \lceil \frac{a}{2} \rceil$. Re-writing Equation (1), when a is even, we get

$$\frac{1-p^a}{(1-p^2)p^a} + A_1 = 3,$$

where $A_1 = \dfrac{p^a + p^{a+2} + \cdots + p^{2a-2} + p^{2b}}{p^a}$ is a natural number. Similarly, when a is odd, we get $\dfrac{1-p^{a+1}}{(1-p^2)p^a} + A_2 = 3$, where $A_2 = \dfrac{p^{a+1} + \cdots + p^{2a-2} + p^{2b}}{p^a}$ is also a natural number. However, both $\dfrac{1-p^a}{(1-p^2)p^a}, \dfrac{1-p^{a+1}}{(1-p^2)p^a}$ are not natural numbers, which leads to a contradiction.

`Case 2:` $b < \lceil \frac{a}{2} \rceil$. Re-writing Equation (1), when a is even, we obtain $\dfrac{1-p^a}{(1-p^2)p^a} + \dfrac{p^{2b}}{p^a} + B_1 = 3$, where $B_1 = \dfrac{p^a + p^{a+2} + \cdots + p^{2a-2}}{p^a} \in \mathbb{N}$. Similarly, when a is odd, we have $\dfrac{1-p^{a+1}}{(1-p^2)p^a} + \dfrac{p^{2b}}{p^a} + B_2 = 3$, where $B_2 = \dfrac{p^{a+1} + \cdots + p^{2a-2}}{p^a} \in \mathbb{N}$. This is a contradiction, similar to the earlier case. □

Generalizing Theorem 1, we have the following result.

Theorem 2. *For a prime p and non-negative integer a,the number p^a is a $[2, \ell]$-near-perfect number if and only if $a = 2$, $\ell = 1$, and $d = 1$.*

Proof. Let $n = p^a$ be a $[2, \ell]$-near-perfect number. By Definition 6, we have

$$1 + p^2 + p^4 + \cdots + p^{2(a-2)} + p^{2(a-1)} - d^2 = \ell p^a. \tag{2}$$

When $a = 1$, we get $n = p$. If $d = 1$, we have $\sigma_2(p) - p^2 - 1 = \ell p$, which implies $\ell p = 0$. This leads to a contradiction. Similarly, we can see that if $d = p$, we get $\sigma_2(p) - p^2 - p^2 = \ell p$, which implies $p^2 + \ell p = 1$. This is again a contradiction, since any $\ell \in \mathbb{N}$.

We now consider $a \geq 2$. If a is even, from Equation (2), we have

$$1 + p^2 + p^4 + \cdots + p^{a-2} + p^a + p^{a+2} + \cdots + p^{2(a-2)} + p^{2(a-1)} - d^2 = \ell p^a.$$

If $d \geq p^{\frac{a}{2}}$, then d^2 is equal to an element of $\{p^a, p^{a+2}, \ldots, p^{2(a-2)}, p^{2(a-1)}\}$. Therefore, we get

$$\frac{1 + p^2 + p^4 + \cdots + p^{a-2}}{p^a} + A = \ell,$$

where $A = \dfrac{p^a + p^{a+2} + \cdots + p^{2(a-2)} + p^{2(a-1)} - d^2}{p^a} \in \mathbb{N}$. On further simplification, we get

$$\frac{1 - p^a}{(1 - p^2)p^a} + A = \ell. \tag{3}$$

However, Equation (3) implies that ℓ is not a natural number, which leads to a contradiction.

If $d < p^{\frac{b}{2}}$, then d^2 is equal to an element of $\{1, p^2, p^4, \ldots, p^{a-4}, p^{a-2}\}$. Therefore, we get

$$\frac{1 + p^2 + p^4 + \cdots + p^{a-2} - d^2}{p^a} + B = \ell,$$

where $B = \dfrac{p^a + \cdots + p^{2(a-1)}}{p^a} \in \mathbb{N}$. On further simplifying, we get

$$\left(\frac{1 - p^a}{1 - p^2} - d^2\right)\frac{1}{p^a} + B = \ell. \tag{4}$$

The first term on the left side of Equation (4) will not be an integer for all values of a and d except for $a = 2$ and $d = 1$. Therefore, $a = 2$ and $d = 1$, and consequently $\ell = 1$.

When a is odd, from Equation (2), we get

$$1 + p^2 + p^4 + \cdots + p^{a-1} + p^{a+1} + \cdots + p^{2(a-2)} + p^{2(a-1)} - d^2 = \ell p^a.$$

If $d \geq p^{\frac{a+1}{2}}$, then d^2 is equal to an element of $\{p^{a+1}, p^{a+3}, \ldots, p^{2(a-1)}\}$. Therefore, we get

$$\frac{1 + p^2 + p^4 + \cdots + p^{a-1}}{p^a} + C = \ell,$$

where $C = \dfrac{p^{a+1} + \cdots + p^{2(a-2)} + p^{2(a-1)} - d^2}{p^a} \in \mathbb{N}$. Upon further simplifying, we get

$$\frac{1 - p^{a+1}}{(1 - p^2)p^a} + C = \ell. \tag{5}$$

But $\dfrac{1 - p^{a+1}}{(1 - p^2)p^a} \notin \mathbb{N}$ and, consequently, from Equation (5) we get $\ell \notin \mathbb{N}$, which is a contradiction. Now if $d < p^{\frac{a+1}{2}}$, then d^2 is equal to an element of $\{1, p^2, p^4 \ldots, p^{a-1}\}$. Therefore, we get

$$\frac{1 + p^2 + p^4 + \cdots + p^{a-1} - d^2}{p^a} + D = \ell,$$

where $D = \dfrac{p^{a+1} + \cdots + p^{2(a-1)}}{p^a} \in \mathbb{N}$. Upon further simplifying, we get

$$\frac{1}{p^a}\left(\frac{1-p^{a+1}}{1-p^2} - d^2\right) + D = \ell. \tag{6}$$

The first term on the left side of Equation (6) will not be an integer for all values of a and d except when $a = 1$ and $d = 1$, but $a = 1$ is not possible, which is a contradiction.

Conversely, let $a = 2, \ell = 1$, and $d = 1$. Then $n = p^2$ and $\sigma_2(p^2) = 1 + p^2 + p^4$. It is easy to see that $\sigma_2(p^2) - p^4 - 1 = p^2$. Therefore, $n = p^a$ is a $[2, \ell]$-near-perfect number when $a = 2, \ell = 1$, and $d = 1$. □

Similarly, generalizing Theorem 1, we have the following result.

Theorem 3. *For a prime p and non-negative integer a, the number p^a is not a $[2, \ell]$-deficient-perfect number for all a except when $a = 2$ and $\ell = 1$.*

The proof follows the same structure as the proof of Theorem 2, so we omit it here.

On further generalizing Theorem 2, we have the following result.

Theorem 4. *For a prime p, the number p^a is a $[k, \ell]$-near-perfect number if and only if $a = k, d = 1$, and $\ell = 1 + p^k + p^{2k} + \cdots + p^{k(k-2)}$.*

Proof. If $n = p^a$ is a $[k, \ell]$-near-perfect number, then

$$1 + p^k + p^{2k} + \cdots + p^{(a-1)k} - d^k = \ell p^a. \tag{7}$$

We may consider two cases.

`Case 1:` $k|a$. Re-arranging Equation (7), we have

$$1 + p^k + p^{2k} + \cdots + p^{(r-1)k} + p^{rk} + p^{(r+1)k} + \cdots + p^{(a-1)k} - d^k = \ell n, \tag{8}$$

where $r = \dfrac{a}{k}$, which means $r < a$. If $d \geq p^{\frac{a}{k}}$, we have $d^k \geq p^a = p^{rk}$. Then d^k is an element from the set $\{p^{rk}, p^{(r+1)k}, \ldots, p^{(a-1)k}\}$. From Equation (8), we have

$$\frac{1 + p^k + p^{2k} + \cdots + p^{(r-1)k}}{p^a} + A = \ell, \tag{9}$$

where $A = \dfrac{p^{rk} + p^{(r+1)k} + \cdots + p^{(a-1)k} - d^k}{p^a} \in \mathbb{N}$. On further simplifying Equation (9), we have

$$\frac{1-p^a}{(1-p^k)p^a} + A = \ell. \tag{10}$$

Here, $\dfrac{1-p^a}{(1-p^k)p^a} \notin \mathbb{N}$, which implies $\ell \notin \mathbb{N}$. This is a contradiction. If $d < p^{\frac{a}{k}}$, then d^k is an element from the set $\{1, p^k, p^{2k} \ldots, p^{(r-1)k}\}$, and from Equation (8), we have

$$\frac{1}{p^a}\left(\frac{1-p^a}{(1-p^k)} - d^k\right) + B = \ell, \tag{11}$$

where $B = \dfrac{p^{rk} + p^{(r+1)k} + \cdots + p^{(a-1)k}}{p^a} \in \mathbb{N}$. The left side of Equation (11) will be an integer only if $k = a$ and $d = 1$. When $k = a$ and $d = 1$, we get

$$\ell = 1 + p^k + p^{2k} + \cdots + p^{(k-2)k}.$$

Case 2: $k \nmid a$. Re-arranging Equation (7), we have

$$1 + p^k + p^{2k} + \cdots + p^{(u-1)k} + p^{uk} + p^{(u+1)k} + \cdots + p^{(a-1)k} = \ell n, \tag{12}$$

where $a = uk + v$, where we clearly have $1 \leq u < a$ and $1 \leq v < a$. Now if $d \geq p^u$, then d^k is an element from the set $\{p^{uk}, p^{(u+1)k}, \ldots, p^{(a-1)k}\}$. From Equation (8), we have

$$\frac{1 + p^k + p^{2k} + \cdots + p^{(u-1)k}}{p^a} + C = \ell, \tag{13}$$

where $C = \dfrac{p^{uk} + p^{(u+1)k} + \cdots + p^{(a-1)k} - d^k}{p^a} \in \mathbb{N}$. On further simplifying Equation (13), we have

$$\frac{1-p^{uk}}{(1-p^k)p^a} + C = \ell, \tag{14}$$

where $\dfrac{1-p^{uk}}{(1-p^k)p^a} \notin \mathbb{N}$. This implies $\ell \notin \mathbb{N}$, which leads to a contradiction. If $d < p^{uk}$, then d^k is an element from the set $\{1, p^k, p^{2k} \ldots, p^{(u-1)k}\}$, and from Equation (12) we have

$$\frac{1}{p^a}\left(\frac{1-p^{uk}}{(1-p^k)} - d^k\right) + D = \ell, \tag{15}$$

where $D = \dfrac{p^{uk} + p^{(u+1)k} + \cdots + p^{(a-1)k}}{p^a} \in \mathbb{N}$. The left side of Equation (15) will not be an integer, as $uk = a - v$. This implies that ℓ is not an integer. This is a contradiction.

Conversely, let $n = p^a$ where $a = k$, $d = 1$, and $\ell = 1 + p^k + p^{2k} + \cdots + p^{k(k-2)}$. Now,

$$\sigma_k(p^k) = 1 + p^k + p^{2k} + \cdots + p^{(k-1)k} + p^{k^2},$$

and

$$\sigma_k(p^k) - n^k - d^k = p^k + p^{2k} + \cdots + p^{(k-1)k}$$
$$= (1 + p^k + p^{2k} + \cdots + p^{(k-2)k})p^k = \ell n.$$

This proves that $n = p^a$ is a $[k, \ell]$-deficient-perfect number. □

Theorem 5. *For a prime p and $k \geq 2$, the number p^a is not a $[k, \ell]$-deficient-perfect number for all a, except when $p = 2$ and $a = 1$. If $k = 1$, then p^a is a $[k, \ell]$-deficient-perfect number if and only if $p = 2, \ell = 1$, and $d = 1$.*

The proof follows the same structure as the proof of Theorem 4, so we will not present it here.

3. Characterization of Positive Integers that Factor into Two or More Prime Powers

To expand the scope of the numbers we worked on in Section 2, in this section we aim to characterize and derive the properties of numbers defined in Section 1 that have two or more prime factors.

Theorem 6. *A natural number $n = 2p$ is not a $[k, \ell]$-near-perfect number, where p is an odd prime.*

Proof. Let $n = 2p$ be a $[k, \ell]$-near-perfect number. Then, the possible values of redundant divisors are $1, 2$, and p. Using Definition 6, we have

$$1 + 2^k + p^k - d^k = 2\ell p.$$

If $d = 1$, we get $2^k + p^k = 2\ell p$, which implies $p^k \equiv 0 \pmod 2$. This is a contradiction as p is odd. If $d = 2$, we get $1 + p^k = 2\ell p$, which implies $p|1$. This is not possible as p is odd. If $d = p$, we get $1 + 2^k = 2\ell p$. This is a contradiction as $1 + 2^k$ is odd. □

Table 1 gives the list of a few $[k, \ell]$-deficient-perfect numbers of the form $2p$ for $1 \leq k \leq 6$.

Theorem 7. *For any non-negative integer a, there is no deficient F-perfect number of the form $n = 2^a p$, where p is any prime, except when $p = 3$ and $a = 1$.*

Proof. Let n be a deficient F-perfect number of the form $n = 2^a p$. Therefore,

$$\sigma_2(n) = \frac{(2^{2a+2} - 1)}{3}(p^2 + 1) = \frac{2^{2a+2}p^2 - p^2 + 2^{2a+2} - 1}{3}.$$

k	p	ℓ	$n = 2p$
1	5	1	10
2	3	3	6
3	17	145	34
4	3 and 11	19 and 667 respectively	6 and 22 respectively
5	5 and 13	319 and 14283 respectively	10 and 26 respectively
6	3 and 43	143 and 73504223 respectively	6 and 86 respectively

Table 1: $[k, \ell]$-deficient-perfect numbers of the form $2p$.

From Definition 1, we have

$$\sigma_2(n) - n^2 + d^2 = 3n.$$

Rearranging, we get

$$d^2 = 3n + n^2 - \sigma_2(n).$$

Therefore,

$$d^2 = 3 \cdot 2^a p + 2^{2a}p^2 - \left(\frac{2^{2a+2}p^2 - p^2 + 2^{2a+2} - 1}{3}\right),$$

which gives us

$$\begin{aligned} 3d^2 &= 3^2 \cdot 2^a p + 3 \cdot 2^{2a}p^2 - 2^{2a+2}p^2 + p^2 - 2^{2a+2} + 1 \\ &= 3^2 \cdot 2^a p + 3 \cdot 2^{2a}p^2 - 4 \cdot 2^{2a}p^2 + p^2 - 2^{2a+2} + 1 \\ &= 3^2 \cdot 2^a p - 2^{2a}p^2 + p^2 - 2^{2a+2} + 1 \\ &= 9 \cdot 2^a p - 2^{2a}p^2 + p^2 - 2^{2a+2} + 1 \\ &= -2^a p(-9 + 2^a p) + p^2 - 2^{2a+2} + 1. \end{aligned}$$

Rearranging the above, we get

$$3d^2 = 1 + p^2 - 2^a p(2^a p - 9) - 2^{2a+2}. \tag{16}$$

By Proposition 1, it is clear that n is not a deficient F-perfect number when $a = 0$. Theorem 1 shows that n is not a deficient F-perfect number when $p = 2$. However, for $a = 1$ and $p = 3$, $n = 6$ is a deficient F-perfect number. So, we now consider the following two cases.

`Case 1:` $a = 1$ and $p > 3$. The possible values of d are $1, 2$, and p. If $d = 1$, Equation (16) becomes $p^2 - 6p + 6 = 0$. This implies p is not a prime, a contradiction. If $d = 2$, Equation (16) gives $p = 3$, which contradicts $p > 3$. If $d = p$, Equation (16) becomes $2p^2 + 5 = 6p$, which leads to a contradiction as $2p^2 + 5$ is odd, but $6p$ is even.

`Case 2:` $a > 1$ and $p > 3$. Depending on the possible values of d, we study the following cases. If $d = 1$, from Equation (16), we get

$$2 + 2^a p(2^a p - 9) + 2^{2a+2} = p^2.$$

This is a contradiction, as $2 + 2^a p(2^a p - 9) + 2^{2a+2}$ is even, but p^2 is odd. If $d = p$, then from Equation (16), we get

$$2p^2 + 2^a p(2^a p - 9) + 2^{2a+2} = 1.$$

This is a contradiction, as $2p^2 + 2^a p(2^a p - 9) + 2^{2a+2}$ is even, but 1 is odd. If $d = 2^b$, where $1 \leq b \leq a$, then from Equation (16), we have

$$3 \cdot 2^{2b} + 2^{2a+2} + p(2^a(2^a p - 9) - p) = 1. \tag{17}$$

But for all $a > 1$ and $p > 3$, we have

$$\begin{aligned} 2^a(2^a p - 9) > 3 \text{ and } 2^a(2^a p - 9) &> p, \\ 2^a(2^a p - 9) - p &> 0, \\ p(2^a(2^a p - 9) - p) &> 0. \end{aligned}$$

It is clear that $3 \cdot 2^{2b} + 2^{2a+2} + p(2^a(2^a p - 9) - p) > 1$, which contradicts Equation (17). If $d = 2^c p$, where $1 \leq c < a$, from Equation (16), we get

$$p(2^a(2^a p - 9) - p) = 1 - 3 \cdot 2^{2a} \cdot p^2 - p^{2a+2}.$$

It is clear that $1 - 3 \cdot 2^{2a} \cdot p^2 - p^{2a+2} < 0$, but $p(2^a(2^a p - 9) - p) > 0$. This is a contradiction. □

Theorem 8. *For a non-negative integer a and a natural number k, a number of the form $n = 2^a p$, where p is an odd prime, is not a $[k, \ell]$-deficient-perfect number when $d = 1$ or $d = p$.*

Proof. Let $n = 2^a p$ be a $[k, \ell]$-deficient-perfect number for $k \geq 1$. When $a = 0$, from Proposition 1, it is clear that n is not a $[k, \ell]$-deficient-perfect number. Let $a > 0$. By using Definition 3, we have

$$\sigma_k(2^a p) - 2^{ak} p^k + d^k = \ell 2^a p,$$

which gives us

$$(1 + 2^k + 2^{2k} + \cdots + 2^{(a-1)k})(1 + p^k) + 2^{ak} + d^k = \ell 2^a p. \tag{18}$$

When $d = 1$ or $d = p$, it is clear that the left side of Equation (18) is not even, which leads to a contradiction since the right side of Equation (18) is even. □

Theorem 9. *For any non-negative integer a, there is no near F-perfect number of the form $n = 2^a p$, where p is any prime number.*

Proof. If $a = 0$ or $p = 2$, it follows from Theorem 1 that n is not a near F-perfect number. Similarly, by using the same theorem, it is clear that n is not a near F-perfect number when $a = 1$ and $p = 2$. If $a = 1$, for any odd prime p, it is obvious from Theorem 6 that n is not a near F-perfect number.

We now consider $n = 2^a p$ to be a near F-perfect number, where p is an odd prime and $a \geq 2$. Using Definition 4, we have

$$1 + 2^2 + 2^4 + \cdots + 2^{2a} + p^2 + 2^2p^2 + \cdots + 2^{2(a-1)}p^2 - d^2 = 3 \cdot 2^a p. \tag{19}$$

We now consider the following cases based on the possible values of d. If $d = 1$, from Equation (19), we get

$$p^2 + 2^2 + 2^4 + \cdots + 2^{2a} + 2^2p^2 + \cdots + 2^{2(a-1)}p^2 = 3 \cdot 2^a p. \tag{20}$$

This is not possible, as the left side of Equation (20) is odd, whereas the right side is even. If $d = p$, Equation (19) becomes

$$1 + 2^2 + 2^4 + \cdots + 2^{2a} + 2^2p^2 + \cdots + 2^{2(a-1)}p^2 = 3 \cdot 2^a p. \tag{21}$$

Again this is not possible, as the left side of Equation (21) is odd, but the right side is even. If $d = 2^b$ in Equation (19), where $1 \leq b \leq a$, then

$$A + 2^{2b}p^2 = 3 \cdot 2^a p,$$

which is equivalent to

$$\frac{A}{p^2 2^{2b} - 3 \cdot 2^a p} + 1 = 0,$$

where $A = (1 + 2^2 + \cdots + 2^{2b-2} + 2^{2b+2} + \cdots + 2^{2a-2})(1 + p^2) + 2^{2a}$. If we assume that $A = 3 \cdot 2^a p - p^2 2^{2b}$, then we have

$$(1 + 2^2 + \cdots + 2^{2b-2} + 2^{2b+2} + \cdots + 2^{2a-2})(1 + p^2) + 2^{2a} = 3 \cdot 2^a p - 2^{2b}p^2,$$

which is equivalent to

$$\begin{aligned}(1 + 2^2 + \cdots + 2^{2b-2} + 2^{2b+2} + \cdots + 2^{2a-4})(1 + p^2)& \\ + 2^{2a} + 2^{2a-2} + 2^a p(2^{a-2}p - 3) + 2^{2b}p^2 = 0.&\end{aligned} \tag{22}$$

But for all $a \geq 2$ we have $2^{a-2}p - 3 \geq 0$. This shows that the left side of Equation (22) is greater than 0, which leads to a contradiction. Therefore, $A \neq 3 \cdot 2^a p - 2^{2b}p^2$, which contradicts $d = 2^b$. If $d = 2^c p$ in Equation (19), where $1 \leq c < a$, then

$$B + 2^{2a} = 3 \cdot 2^a p,$$

which is equivalent to

$$\frac{B}{2^{2a} - 3 \cdot p \cdot 2^{a}} + 1 = 0,$$

where $B = (1 + 2^2 + \cdots + 2^{2c-2} + 2^{2c+2} + \cdots + 2^{2a-2})(1 + p^2) + 2^{2c}$. Let us assume that $B = 3 \cdot 2^a \cdot p - 2^{2a}$. Therefore,

$$(1 + 2^2 + \cdots + 2^{2c-2} + 2^{2c+2} + \cdots + 2^{2a-2})(1 + p^2) + 2^{2c} = 3 \cdot 2^a \cdot p - 2^{2a},$$

which give us

$$(1 + 2^2 + \cdots + 2^{2c-2} + 2^{2c+2} + \cdots + 2^{2a-4})(1 + p^2) \\ + 2^{2c} + 2^{2a-2} + 2^a p(2^{a-2}p - 3) + 2^{2a} = 0. \quad (23)$$

But for all $a > 1$, we have $2^{a-2}p - 3 \geq 0$, this shows that the left side of Equation (23) is greater than 0. This is a contradiction. Therefore, $B \neq 3 \cdot 2^a \cdot p - 2^{2a}$, contradicting $d = 2^c p$. □

In the next theorem, we generalize Theorem 9 from near F-perfect number to $[2, \ell]$-near-perfect number.

Theorem 10. *For an odd prime p and a non-negative integer a, a number $n = 2^a p$ is not a $[2, \ell]$-near-perfect number.*

Proof. Let $n = 2^a p$ be a $[2, \ell]$-near-perfect number, where p is an odd prime. When $a = 0$, from Proposition 2, it is clear that n is not a $[2, \ell]$-near-perfect number. When $a = 1$, from Theorem 6, it is clear that n is not a $[k, \ell]$-near-perfect number for $k = 2$. Let $a > 1$. From Definition 6, we have

$$(1 + 2^2 + 2^4 + \cdots + 2^{2a})(1 + p^2) - 2^{2a}p^2 - d^2 = 2^a p\ell.$$

Since $a > 1$,

$$2^a p\ell - (2^2 + 2^4 + \cdots + 2^{2a})(1 + p^2) + 2^{2a}p^2 \equiv 0 \pmod 4.$$

Therefore, $1 + p^2 - d^2 \equiv 0 \pmod 4$. If $d = 1$, then $p^2 \equiv 0 \pmod 4$. This is a contradiction as p is an odd prime. If $d = p$, then $1 \equiv 0 \pmod 4$, which is not possible. If $2|d$, then $1 + p^2 \equiv 0 \pmod 4$. This is a contradiction as $1 \equiv 1 \pmod 4$ and $p^2 \equiv 0, 1 \pmod 4$. Therefore, $1 + p^2 \equiv 1, 2 \pmod 4$. □

The next theorem is an immediate generalization from $[2, \ell]$-near-perfect number to $[k, \ell]$-near-perfect number for any natural number k.

Theorem 11. *For an odd prime p and $a > 1$, the positive integer $n = 2^a p$ is not a $[k, \ell]$-near-perfect number when $d = 1$ and $d = p$.*

Proof. Let $n = 2^a p$ be a $[k, \ell]$-near-perfect number where $a > 1$. We know that,

$$\sigma_k(2^a p) = (1 + 2^k + 2^{2k} + 2^{3k} + \cdots + 2^{ak})(1 + p^k).$$

From Definition 6, we have

$$\sigma_k(n) - n^k - d^k = \ell n.$$

Therefore,

$$(2^k + 2^{2k} + 2^{3k} + \cdots + 2^{ak})(1 + p^k) + 1 + p^k - 2^{ak}p^k - d^k = \ell 2^a p. \tag{24}$$

We now consider two cases for the value of k.

`Case 1:` $k = 1$. From Equation (24), we have

$$(2^2 + 2^3 + \cdots + 2^a)(1 + p) - 2^a p - \ell 2^a p = d - 3 - 3p.$$

Since $a > 1$, we see that

$$(2^2 + 2^3 + \cdots + 2^a)(1 + p) - 2^a p - \ell 2^a p \equiv 0 \pmod 4.$$

This implies,

$$d - 3 - 3p \equiv 0 \pmod 4.$$

Therefore, when d=1, we have $-(2+3p) \equiv 0 \pmod 4$. This is a contradiction, since $2 + 3p$ is an odd number. When $d = p$, we have $-(3 + 2p) \equiv 0 \pmod 4$. This is a contradiction as $3 + 2p$ is an odd number.

`Case 2:` $k > 1$. From Equation (24), we have

$$2^{ak}p^k + \ell 2^a p - (2^k + 2^{2k} + 2^{3k} + \cdots + 2^{ak})(1 + p^k) \equiv 0 \pmod 4.$$

This implies

$$1 + p^k - d^k \equiv 0 \pmod 4.$$

Since $a > 1$, for $d = 1$, we have $p^k \equiv 0 \pmod 4$. This is a contradiction as p is an odd prime. For $d = p$, we have $1 \equiv 0 \pmod 4$. This is a contradiction.

Therefore, $n = 2^a p$ is not a $[k, \ell]$-near-perfect number when $d = 1$ and $d = p$ for $a > 1$. □

Theorem 12. *For any non-negative integer a and any prime p, the only deficient F-perfect number of the form $n = 2p^a$ is 6.*

Proof. When $a > 1$, using Definition 1, we have

$$\sigma_2(n) - n^2 + d^2 = 3n.$$

Since

$$\sigma_2(2p^a) = (1+2^2)(1+p^2+p^4+\cdots+p^{2a}) \\ = 5+5p^2+5p^4+\cdots+5p^{2a},$$

applying Definition 1, we have

$$6p^a = 5+5p^2+5p^4+\cdots+5p^{2(a-1)}+5p^{2a}-2^2p^{2a}+d^2 \\ = 5+5p^2+5p^4+\cdots+5p^{2(a-1)}+p^{2a}+d^2,$$

which implies

$$5+5p^2+5p^4+\cdots+5p^{2(a-1)}+d^2+p^a(p^a-6)=0.$$

Since $a>1$ and p is an odd prime, $p^a \geq 9$, and therefore, $(p^a-6)>0$. This shows that $5+5p^2+5p^4+\cdots+5p^{2(a-1)}+d^2+p^a(p^a-6)>0$. Therefore, if $a>1$, $n=2p^a$ is not a deficient F-perfect number.

Finally, for $a=1$, we have $n=2p$ and $\sigma_2(2p)=6p+4p^2-d^2$. Therefore,

$$d^2 = 6p-p^2-5. \tag{25}$$

It can easily be observed that the only prime number that satisfies Equation (25) is $p=3$ when $d=2$. Hence, the only deficient F-perfect number of the form $n=2p^a$ is 6. □

Theorem 13. *For any nonnegative integer a, there is no near F-perfect number of the form $n=2p^a$, where p is any prime.*

Proof. Let $n=2p^a$ be a positive integer. If p is even, it follows from Theorem 1 that n cannot be a near F-perfect number. Therefore, p is an odd prime. When $a=0$, again from Theorem 1, we see that n is not a near F-perfect number. When $a=1$, it follows from Theorem 1 that n is not a near F-perfect number. Therefore, $a>1$. Let $n=2p^a$ be a near F-perfect number, where $a>1$ and p is an odd prime. From Definition 4, we have

$$5+5p^2+5p^4+\cdots+5p^{2(a-1)}+p^{2a}=6p^a+d^2. \tag{26}$$

We now consider the following four possible values of d.

`Case 1:` $d=1$. From Equation (26), we get

$$5+5p^2+5p^4+\cdots+5p^{2(a-1)}+p^{2a}=6p^a+1.$$

This is a contradiction, as

$$5+5p^2+5p^4+\cdots+5p^{2(a-1)}+p^{2a}>6p^a+1.$$

Case 2: $d = 2$. From Equation (26), we get

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(a-1)} + p^{2a} = 6p^a + 4.$$

This is a contradiction, as

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(a-1)} + p^{2a} > 6p^{2a} + 4.$$

Case 3: $d = 2p^q$, where $2 \leq q < a$. From Equation (26), it is clear that

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(a-1)} + p^{2a} = 6p^{2a} + 4p^{2q}.$$

Since $q < a$, there exists some $s = q$, such that

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(s-1)} + 5p^{2s} + 5p^{2(s+1)} \cdots + 5p^{2(a-1)} + p^{2a} = 6p^{2a} + 4p^{2q}.$$

This is a contradiction, as

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(s-1)} + 5p^{2s} + 5p^{2(s+1)} \cdots + 5p^{2(a-1)} + p^{2a} > 6p^{2a} + 4p^{2q}.$$

Case 4: $d = p^b$, where $1 < b \leq a$. If $1 < b < a$ from Equation (26), we see that

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(b-1)} + 5p^{2b} + 5p^{2(b+1)} \cdots + 5p^{2(a-1)} + p^{2a} = 6p^{2a} + 4p^{2b}.$$

This is a contradiction, as

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(b-1)} + 5p^{2b} + 5p^{2(b+1)} \cdots + 5p^{2(a-1)} + p^{2a} > 6p^{2a} + 4p^{2b}.$$

If $b = a$, from Equation (26), we get

$$5 + 5p^2 + 5p^4 + \cdots + 5p^{2(a-1)} + p^{2a} = 6p^a + p^{2a},$$

that is,

$$5 + 5p^2 + 5p^4 + \cdots + \frac{p^a(5p^a - 6p^2)}{p^2} = 0.$$

For all $a > 2$, we see that $\dfrac{p^a(5p^a - 6p^2)}{p^2} > 0$ and therefore,

$$5 + 5p^2 + 5p^4 + \cdots + \frac{p^a(5p^a - 6p^2)}{p^2} > 0.$$

This is a contradiction. When $a = 2$, from Equation (26), we get

$$5 + 5p^2 + 5p^4 - 4p^4 - p^4 = 6p^2.$$

This implies

$$5 = p^2.$$

This is a contradiction as p is an odd prime. Therefore, there is no near F-perfect number of the form $n = 2p^a$, where p is any prime. □

Theorem 14. *For any non-negative integer a and an odd prime p, let us consider a number of the form $n = 2p^a$. Then*

1. *The only $[k, \ell]$-near-perfect number when $k = 1$ is 18.*
2. *If n is a $[k, \ell]$-near-perfect number, then p is a solution of the congruence $1 + 2^k \equiv 0 \pmod{p^2}$ when $k > 1$.*

Furthermore, a is even if and only if d is odd.

Proof. If $a = 1$, then from Theorem 6, it is clear that n is not a $[k, \ell]$-near-perfect number. Therefore, let $n = 2p^a$ be a $[k, \ell]$-near-perfect number for $a > 1$. By Definition 6, we have

$$(1 + 2^k)(1 + p^k + p^{2k} + \cdots + p^{ak}) - 2^k p^{ak} - d^k = 2\ell p^a. \tag{27}$$

First we prove Statement 1. If $k = 1$, from Equation (27), we get

$$3(1 + p) - d = 2\ell p^a - 3(p^2 + p^4 + \cdots + p^a) + 2p^a.$$

As

$$2\ell p^a - 3(p^2 + p^4 + \cdots + p^a) + 2p^a \equiv 0 \pmod{p^2},$$

this implies that

$$3(1 + p) - d \equiv 0 \pmod{p^2}.$$

When the value of d is 1 or 2, the congruence is invalid. When $d = p$, we get $3 + 2p \equiv 0 \pmod{p^2}$, which implies that $p = 3$. Therefore, from Equation (27), we get

$$3^2(1 + 3 + 3^2 + \cdots + 3^{a-1}) - 2 \cdot 3^a = \ell 2 \cdot 3^a,$$

from which, after simplification, we obtain

$$\frac{9(3^a - 1)}{4 \cdot 3^a} - 1 = \ell. \tag{28}$$

For all a, except when $a = 2$, it is easy to see that the first term on the left side of Equation (28) is not a natural number, which implies $\ell \notin \mathbb{N}$. This is a contradiction. When $a = 2$, it is clear that $\ell = 1$. Therefore, when $a = 2$ and $p = 3$, it is easy to see that 18 is a $[1, 1]$-near-perfect number. This proves the first statement.

Now we prove statement 2. When $k > 1$ in Equation (27), we have

$$1 + 2^k - d^k = 2\ell p^a - (1 + 2^k)(p^k + p^{2k} + \cdots + p^{ak}) + 2^k + p^{ak}.$$

As

$$2\ell p^a - (1 + 2^k)(p^k + p^{2k} + \cdots + p^{ak}) + 2^k + p^{ak} \equiv 0 \pmod{p^2},$$

this implies that $1+2^k-d^k \equiv 0 \pmod{p^2}$. If $p^2 \nmid d^k$, then $d^k = 1$ *or* 2^k. In both cases, the congruence is invalid. If $p^2|d^k$, then from Equation (27), we get $1+2^k \equiv 0 \pmod{p^2}$. Also, from Equation (27), it is easy to see that a is even when d is odd. □

Table 2 gives a list of $[k,\ell]$-near-perfect numbers of the form $2p^a$ for $k = 1, 3, 9, 10, 15$.

k	$1+2^k$	n	d	ℓ
1	$1+2^1=3$	$2\cdot 3^2$	3	1
3	$1+2^3=3^2$	$2\cdot 3^2$	3	53
			3^2	14
9	$1+2^9=3^3\cdot 19$	$2\cdot 3^2$	3	22083261
			3^2	560994
9	$1+2^9=3^3\cdot 19$	$2\cdot 3^3$	$2\cdot 3$	144895263260
			$2\cdot 3^2$	141222129692
10	$1+2^{10}=5^2\cdot 41$	$2\cdot 5^2$	5	1907548632833
			2	200195333
15	$1+2^{15}=3^2\cdot 11\cdot 331$	$2\cdot 3^2$	3	11464517612333
			3^2	26122187014

Table 2: $[k,\ell]$-near-perfect numbers of the form $2p^a$.

Theorem 15. *For a non-negative integer a, there is no near F-perfect number of the form 2^ap^b, when $2|b$.*

Proof. Let $n = 2^ap^b$ be a near F-perfect number where $a, b \in \mathbb{N}$ and $b \equiv 0 \pmod 2$. If $p = 2$, n is not a near F-perfect number by Theorem 1. Similarly, by using the same theorem, we can show that n is not a near F-perfect number if $a = 0$ and $b \geq 2$. By Theorem 13, we can show that if $a = 1$ and $b \geq 1$, then n is not a near F-perfect number. Therefore, for an odd prime p, we consider $n = 2^ap^b$ to be a near F-perfect number, where $a > 1$ and $b \geq 2$ with $2|b$. Since n is a near F-perfect number, we have

$$\sigma_2(2^ap^b) - 2^{2a}p^{2b} - d^2 = 3\cdot 2^ap^b, \tag{29}$$

where

$$\sigma_2(2^ap^b) = \frac{2^{2(a+1)}-1}{3}(1+p^2+p^4+\cdots+p^{2b}). \tag{30}$$

Since $2|b$, from Equation (30), we see that $\sigma_2(2^ap^b)$ is odd. From Equation (29), it is clear that the redundant divisor d is odd. Let $d = p^c$, where $0 < c \leq b$. Then, from Equation (29), we have

$$\frac{2^{2a+2}-1}{3}\frac{p^{2b+2}-1}{p^2-1} - 2^{2a}p^{2b} - p^{2c} = 3\cdot 2^ap^b.$$

Simplifying further, we obtain

$$2^{2a}p^{2b+2} + 3 \cdot 2^{2a}p^{2b} + 3 \cdot 2^a p^b - 3^2 \cdot 2^a p^{b+2} - 2^{2a+2} - p^{2b+2} < 3p^{2c+2},$$

which eventually reduces to

$$p^{2b+2}(2^{2a} - 1) + A < 3p^{2c+2} + 2^{2a+2}, \tag{31}$$

where

$$A = 3 \cdot 2^{2a}p^{2b} - 3^2 2^a p^{b+2} + 3 \cdot 2^a p^b.$$

Since $p^c \leq p^b$ and $(2^{2a} - 1) > 3$ for all $a \geq 2$, we have

$$p^{2b+2}(2^{2a} - 1) > 3p^{2c+2}. \tag{32}$$

For all $b \geq 2$, we have $p^{2b} \geq p^{b+2}$ and $p^b > 4$. Therefore, $3p^{2b} - 4 > 3p^{b+2} - p^b$. Since for all $a \geq 2$, we have $2^a > 3$. This implies,

$$2^a(3p^{2b} - 4) > 3(3p^{b+2} - p^b).$$

On further simplification, we get

$$3 \cdot 2^a p^{2b} - 2^{a+2} > 3^3 p^{b+2} - 3p^b,$$

which implies

$$3 \cdot 2^{2a}p^{2b} - 3^2 2^a p^{b+2} + 3 \cdot 2^a p^b > 2^{2a+2}.$$

Therefore, we see that

$$A > 2^{2a+2}. \tag{33}$$

Thus, by using Equation (32) and (33), it is easy to see that Equation (31) leads to a contradiction. Hence, we conclude that there is no near F-perfect number of the form $2^a p^b$ when $2|b$. □

Theorem 16. *For any two odd primes p_1 and p_2, there is no deficient F_k-perfect number of the form $n = p_1^a p_2$, where a is a non-negative integer.*

Proof. Let $n = p_1^a p_2$ be a deficient F_k-perfect number. Using Definition 2, we have

$$(1 + p_1^k + p_1^{2k} + \cdots + p_1^{ak})(1 + p_2^k) - p_1^{ak}p_2^k + d^k = 3p_1^a p_2. \tag{34}$$

Since both p_1 and p_2 are odd, d is odd and therefore, the left side of Equation (34) is even, but the right side of Equation (34) is odd. This is a contradiction. □

Theorem 17. *For any two odd primes p_1 and p_2, there are no near F_k-perfect numbers of the form $n = p_1^a p_2$, where a is a non-negative integer.*

The proof follows the same structure as the proof of Theorem 16, so we omit it here.

Theorem 18. *For a non-negative integer a, there is no deficient F_k-perfect number of the form $2p^a$ when $k > 2$ and p is an odd prime.*

Proof. When $a = 0$, by Proposition 1 it is clear that n is not a deficient F_k-perfect number. Let $a > 1$. When $k = 2$, it is clear from Theorem 12 that the only deficient F_k-perfect number is 6. When $k > 2$, let $n = 2p^a$ be a deficient F_k-perfect number. Using Definition 2, we have

$$(1 + 2^k) + (1 + 2^k)p^k + \cdots + (1 + 2^k)p^{k(a-1)} + p^{ak} + d^k = 6p^a.$$

When $a \geq 1$ and $k \geq 3$, as $p \geq 3$, we have $1 + 2^k + p^{ak} \geq 36$ and $6p^a \geq 18$. This implies that $1+2^k+p^{ak} > 6p^a$. Therefore, $1+2^k+p^{ak}-6p^a > 1$, which contradicts

$$(1 + 2^k) + (1 + 2^k)p^k + \cdots + (1 + 2^k)p^{k(a-1)} + p^{ak} + d^k - 6p^a = 0$$

for all values of d. Therefore, we conclude that there is no deficient F_k-perfect number of the form $2p^a$ when $k > 2$ and p is an odd prime. □

Theorem 19. *For natural numbers a and b, and odd primes p_1 and p_2, there is no odd deficient F_k-perfect number of the form $n = p_1^a p_2^b$ when $a + b \equiv 1 \pmod 2$ or both a and b are odd.*

Proof. Let $n = p_1^a p_2^b$ be an odd deficient F_k-perfect number, where $a+b \equiv 1 \pmod 2$ or both $a, b \equiv 1 \pmod 2$. Here, p_1 and p_2 are odd primes, with $a, b \in \mathbb{N}$. Using Definition 2, we get

$$\sigma_k(p_1^a p_2^b) - p_1^{ka} p_2^{kb} + p_1^{ka_1} p_2^{kb_1} = 3p_1^a p_2^b. \tag{35}$$

Where $d = p_1^{a_1} p_2^{b_1}$, and $a_1 \leq a, b_1 \leq b$, and $a_1 + b_1 < a + b$. The condition $a + b \equiv 1 \pmod 2$ leads to two possibilities, either $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$, or $a \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 2$. When $a \equiv 1 \pmod 2$ and $b \equiv 0 \pmod 2$, we have

$$\sigma_k(p_1^a p_2^b) = (1 + p_1^k + p_1^{2k} + p_1^{3k} \cdots + p_1^{ka})(1 + p_2^2 + p_2^{2k} + p_2^{3k} \cdots + p_2^{kb}) \equiv 0 \pmod 2.$$

This implies that the left side of Equation (35) is even and the right side of Equation (35) is odd, which is not possible. Therefore, this leads to a contradiction. In the same way, we can prove for $a \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 2$, and $a, b \equiv 1 \pmod 2$. □

Theorem 20. *For natural numbers a and b and odd primes p_1 and p_2, there is no odd near F_k-perfect number of the form $n = p_1^a p_2^b$ where $a + b \equiv 1 \pmod 2$ or if both a and b are odd.*

The proof follows the same structure as the proof of Theorem 19, so we omit it here.

Theorem 21. *For $m > 0$ and $1 \leq i \leq m$, there is no deficient F_k-perfect number of the form $n = p_1p_2p_3 \cdots p_m$, where the p_i's are odd primes.*

Proof. Let $n = p_1p_2p_3 \cdots p_m$ be a deficient F_k-perfect number, where the p_i's are odd primes and $m > 0$. When $m = 1$, it is clear from Proposition 1 that n is not a deficient F_k-perfect number. Therefore, let $m > 1$, from which we know that

$$\sigma_k(p_1p_2p_3 \cdots p_m) = (1 + p_1^k)(1 + p_2^k)(1 + p_3^k) \cdots (1 + p_m^k). \tag{36}$$

From Definition 2, we have

$$(1 + p_1^k)(1 + p_2^k)(1 + p_3^k) \cdots (1 + p_m^k) - p_1^kp_2^kp_3^k \cdots p_m^k - d^k = 3p_1p_2p_3 \cdots p_m. \tag{37}$$

Thus, it is clear that $\sigma_k(p_1p_2p_3 \cdots p_m)$ is even in Equation (36). This implies that the left side of Equation (37) is even. However, the right side of Equation (37) is odd, leading to a contradiction. □

Theorem 22. *For $m > 0$ and $1 \leq i \leq m$, there is no linear near F_k-perfect number of the form $n = p_1p_2p_3 \cdots p_m$, where the p_i's are odd primes.*

The proof follows the same structure as the proof of Theorem 21, so we omit it here.

4. Concluding Remarks

In this paper, the concept of F-perfect numbers is extended to two new types of numbers called near and deficient F-perfect numbers. These numbers are characterized and further generalized to near and deficient F_k-perfect numbers, whose properties are explored. Finally, we further generalized these numbers to $[k, l]$-near and $[k, l]$-deficient-perfect numbers and we characterized these numbers having one and two prime factors. Further work may focus on determining bounds for these numbers and characterizing $[k, l]$-near (deficient)-perfect numbers with more than two prime factors.

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