#A88 INTEGERS 25 (2025)

ON MONOCHROMATIC SOLUTIONS OF LINEAR EQUATIONS USING AT LEAST THREE COLORS

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Received: 2/3/25, Revised: 6/16/25, Accepted: 9/6/25, Published: 9/17/25

Abstract

We study the number of monochromatic solutions to linear equations in $\{1, \ldots, n\}$ when we color the set by at least three colors. We consider the r-commonness for $r \geq 3$ of linear equations with an odd number of terms and also prove that any 2-uncommon equation is r-uncommon over the integers for any $r \geq 3$.

1. Introduction

1.1. Background and Motivation

In 1996, Graham, Rödl, and Ruciński [4] asked about the minimum number of monochromatic solutions to the Schur equation $x + y = z$ with $(x, y, z) \in [n]^3$, where the set $[n] := \{1, \ldots, n\}$ is 2-colored. Robertson and Zeilberger [5] showed that the minimum number of monochromatic Schur triples in a 2-coloring of $[n]$ is asymptotically $n^2/11 + O(n)$. This is less than $(1/8 + o(1))n^2$, which is the expected number of monochromatic solutions under uniformly random 2-colorings.

One can ask a similar question for more general linear equations

$$a_1x_1 + \cdots + a_kx_k = 0, \quad a_1, \ldots, a_k \in \mathbb{Z}\backslash\{0\}, k \in \mathbb{Z}_{\geq 3}.$$

This problem has been studied for several linear equations, including generalized Schur triples, K-term arithmetic progressions, and constellations. In fact, it still remains unknown which linear equations have uniformly random 2-colorings that asymptotically minimize the number of monochromatic solutions.

We define a k-term linear equation to be 2-*common over the integers* if any 2-coloring of $[n]$ has at least as many monochromatic solutions asymptotically (as $n \to \infty$) as uniformly random colorings. Otherwise, we say that the equation is 2-*uncommon over the integers.* More generally, if we change 2 to any positive integer

DOI: 10.5281/zenodo.17144487

r greater than 1, we can define linear equations to be r-common over the integers in a similar fashion.

Recently, Costello and Elvin [1] showed that all 3-term equations are 2-uncommon over the integers. In the same paper, they conjectured that an equation is 2-common over the integers if and only if the number of terms is even and the equation has a canceling partition. We say that the linear equation

$$a_1x_1 + \cdots + a_kx_k = 0 \tag{1}$$

has a *canceling partition* if we can partition the coefficients into pairs $\{a_i, a_j\}$ such that $a_i + a_j = 0$. Clearly, if a canceling partition exists, then k must be even.

More recently, Dong, Mani, Pham, and Tidor [2] showed that the conjecture is false by showing that the linear equation $x_1 + 2x_2 - x_3 - 2x_4 = 0$ is 2-uncommon over the integers. The case when the number of terms in Equation (1) is odd and at least five still remains unknown.

While commonness over the integers is still a mystery, Versteegen [8] proved that an equation is 2-uncommon over a finite abelian group G, with the order of G coprime to every coefficient of the equation if and only if k is even and the equation has no canceling partition. This generalizes the result of [3], where the same result over finite fields was proved. See also [6] for an introduction to the topic.

We also note that another motivation for considering r-commonness of a linear equation is because of the Sidorenko property of an equation, which was first introduced in [6]. The notion is inspired by Sidorenko's conjecture on graphs [7], which is still a major open problem in extremal graph theory. We recall the definition here. Given a linear equation $L : a_1x_1 + \cdots + a_kx_k = 0$ over a finite abelian group G, if $\mathcal{C}(L)$ denotes the number of solutions of $L = 0$ in G^k, we say that L is *Sidorenko in* G if for every $A \subseteq G$ we have

$$t_L(1_A) \geq \left(\frac{|A|}{|G|}\right)^k$$

where 1_A is the indicator function of A and

$$t_L(1_A) := \frac{1}{|\mathcal{C}(L)|} \sum_{v \in \mathcal{C}(L)} \prod_{i=1}^{k} 1_A(v_i).$$

Clearly, if an equation is Sidorenko over G, then it is r-common for every $r \geq 2$. The results for finite fields are given in [3], and for general finite abelian groups in [8]. One can think of the notion of r-commonness for $r \geq 3$ as being the case between Sidorenko and 2-common, which is often referred to simply as 'common' in the existing literature.

1.2. Main Results

We now explore the phenomenon of r-commonness for linear equations with $r \geq 3$. We first consider the r-commonness over the integers for k-term linear equations where $k > 1$ is an odd positive integer and $r > 2$. This problem is easier than in the case of 2-commonness. In fact, all such equations are r-uncommon over the integers as a corollary of the following theorem, whose proof will appear in Section 3.

Theorem 1. *Let G be an arbitrary nontrivial finite abelian group and let E be an arbitrary linear equation $a_1x_1 + \cdots + a_{2m+1}x_{2m+1} = 0$ such that $a_i \in \mathbb{Z}\backslash\{0\}$ for each $i = 1, \ldots, 2m + 1$ with $m \geq 1$. Assume $|G|$ is coprime to each a_i for $i = 1, \ldots, 2m + 1$. Then the equation E is r-uncommon over G for every $r \geq 3$.*

Corollary 1. *Let $m \geq 1$ be an integer and let E be a $(2m + 1)$-linear equation $a_1x_1 + \cdots + a_{2m+1}x_{2m+1} = 0$, where $a_i \in \mathbb{Z}\backslash\{0\}$ for each $i = 1, \ldots, 2m + 1$. Then E is r-uncommon over the integers for every $r \geq 3$.*

Next, we consider the case where the number of terms in the equation is even. As mentioned, it is known from [8] that every equation that has no canceling partition is 2-uncommon over an abelian group G, provided the order of G is coprime to every a_i. In particular, by Lemma 1, such an equation is 2-uncommon over the integers. As noted earlier, we have that the equation $x_1 + 2x_2 - x_3 - 2x_4 = 0$ is 2-uncommon over the integers. We have that these equations are also r-uncommon, using a different proof method than in Corollary 1, as shown in Section 3.

Theorem 2. *Let $k \geq 3$ and let E be a 2-uncommon equation over the integers $a_1x_1 + \cdots + a_kx_k = 0$ with $a_i \in \mathbb{Z}\backslash\{0\}$ for each $i = 1, \ldots, k$. Then the equation is also r-uncommon over the integers for any $r \geq 3$.*

Corollary 2. *Every linear equation $a_1x_1 + \cdots + a_{2m}x_{2m} = 0, m \geq 2$, that has no canceling partition is r-uncommon over the integers for any $r \geq 2$. The same is true for the equation $x + 2y - z - 2w = 0$.*

2. Notation and Conventions

We now introduce some notation that we will use throughout this paper. We write $f = O(g)$ or $f \ll g$ if there exists a constant C such that $|f| \leq Cg$. We denote a prime number by p, and a cyclic group of order ℓ is denoted by $\mathbb{Z}/\ell\mathbb{Z}$ where $\ell > 1$ is a positive integer. Let $f, g : G \to [0, 1]$, which we interpret as probabilistic colorings via

$$f(t) = \mathbb{P}[t \text{ is the first color}]$$
$$g(t) = \mathbb{P}[t \text{ is the second color}].$$

One may think that we use red, green, and blue as colors, with red being the first color, green the second color, and blue the third color. We define the *Fourier transform of* f, denoted by $\hat{f}$, by

$$\hat{f}(\xi) := \frac{1}{|G|}\sum_{t\in G} f(t)\,\mathbf{e}(-\xi\cdot t)$$

where $\mathbf{e}(x) = e^{2\pi i x}$ and ξ is a homomorphism from $\widehat{G}$ to $\mathbb{R}/\mathbb{Z}$ acting as $\xi : t \mapsto \xi\cdot t$. Here $\widehat{G}$ denotes the dual group of G, which is isomorphic to G. The Fourier transform of g, denoted by $\hat{g}$, is defined similarly to $\hat{f}$.

We can write the expected number of red solutions of $a_1x_1+\cdots+a_{2m+1}x_{2m+1} = 0$ over G in terms of Fourier transforms:

$$\mathbb{E}[\text{number of red solutions}] = |G|^{2m}\sum_{t\in G}\hat{f}(a_1t)\ldots\hat{f}(a_{2m+1}t).$$

Note that the formula above holds only if at least one of $a_1,\ldots,a_{2m+1}$ is coprime to $|G|$, which we always assume. The expected proportion of monochromatic solutions in G is given by

$$\begin{aligned}&\mu_{a_1x_1+\cdots+a_{2m+1}x_{2m+1}=0}(f,g)\\ &=\sum_{t\in G}\hat{f}(a_1t)\ldots\hat{f}(a_{2m+1}t)+\sum_{t\in G}\hat{g}(a_1t)\ldots\hat{g}(a_{2m+1}t)\\ &\quad+\sum_{t\in G}(\widehat{1-f-g})(a_1t)\ldots(\widehat{1-f-g})(a_{2m+1}t).\end{aligned}\tag{2}$$

3. Linear Equations That Are r-Uncommon

We begin by proving Theorem 1. Corollary 1 follows from Theorem 1 and the following lemma, which appears as [1, Lemma 2.1]. For an arbitrary set S and an arbitrary linear equation $E : a_1x_1 + \cdots + a_kx_k = 0$, we denote by $\mu(S)$ to be the proportion of the minimum number of monochromatic solutions relative to the total number of solutions of E in S^k.

Lemma 1. *Let E be a linear equation $a_1x_1+\cdots+a_{2m+1}x_{2m+1} = 0$. Then we have*

$$\limsup_{n\to\infty}\mu_E([n]) \le \mu_E(\mathbb{Z}/\ell\mathbb{Z})$$

for any positive integer ℓ.

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. Let E be a linear equation $a_1x_1+\cdots+a_{2m+1}x_{2m+1} = 0$ where $a_i \in \mathbb{Z}\backslash\{0\}$. We first prove that the equation is 3-uncommon and then show that it is also r-uncommon for any $r > 3$ by generalizing the case where $r = 3$. We need to find f and g such that the quantity in (2) is less than $\frac{1}{3^{2m}}$. But we have

$$\begin{aligned}
&\mu_{a_1x_1+\cdots+a_{2m+1}x_{2m+1}=0}(f,g)\\
&= \frac{1}{3^{2m}} + \sum_{t\in G\setminus\{0\}} \hat{f}(a_1t)\dots\hat{f}(a_{2m+1}t) + \sum_{t\in G\setminus\{0\}} \hat{g}(a_1t)\dots\hat{g}(a_{2m+1}t)\\
&\quad + \sum_{t\in G\setminus\{0\}} (\widehat{1-f-g})(a_1t)\dots(\widehat{1-f-g})(a_{2m+1}t)\\
&= \frac{1}{3^{2m}} + \sum_{t\in G\setminus\{0\}} \hat{f}(a_1t)\dots\hat{f}(a_{2m+1}t) + \sum_{t\in G\setminus\{0\}} \hat{g}(a_1t)\dots\hat{g}(a_{2m+1}t)\\
&\quad + \sum_{t\in G\setminus\{0\}} (-\hat{f}-\hat{g})(a_1t)\dots(-\hat{f}-\hat{g})(a_{2m+1}t).
\end{aligned}$$

The last equality follows from the fact that for $s \neq 0$, we have $(\widehat{1-f})(s) = -\hat{f}(s)$. We refer to the quantity

$$\begin{aligned}
\sum_{t\in G\setminus\{0\}} \hat{f}(a_1t)\dots\hat{f}(a_{2m+1}t) &+ \sum_{t\in G\setminus\{0\}} \hat{g}(a_1t)\dots\hat{g}(a_{2m+1}t)\\
&+ \sum_{t\in G\setminus\{0\}} (-\hat{f}-\hat{g})(a_1t)\dots(-\hat{f}-\hat{g})(a_{2m+1}t)
\end{aligned}$$

as the *deviation*.

We assume without loss of generality that $\hat{f}(0) = \hat{g}(0) = \frac{1}{3}$, which is equivalent to requiring that red, green, and blue appear with equal overall probability. To ensure that the proportion is less than that of a uniformly random coloring, it suffices to find f and g such that $\hat{f}(0) = \hat{g}(0) = \frac{1}{3}$ and the deviation is negative. Therefore, it is enough to find f and g such that $\hat{f}(0) = \hat{g}(0) = \frac{1}{3}$ and

$$\begin{aligned}
\sum_{t\in G\setminus\{0\}} \hat{f}(a_1t)\cdots\hat{f}(a_{2m+1}t) &+ \sum_{t\in G\setminus\{0\}} \hat{g}(a_1t)\cdots\hat{g}(a_{2m+1}t)\\
&+ \sum_{t\in G\setminus\{0\}} (-\hat{f}-\hat{g})(a_1t)\cdots(-\hat{f}-\hat{g})(a_{2m+1}t) < 0.
\end{aligned}$$

By Fourier inversion, f and g are uniquely determined by their Fourier coefficients. First, we note that f and g are real-valued if and only if $\hat{f}$ and $\hat{g}$ are Hermitian, i.e., $\overline{\hat{f}(s)} = \hat{f}(-s)$ and $\overline{\hat{g}(s)} = \hat{g}(-s)$. So we need to ensure this condition holds for $\hat{f}$ and $\hat{g}$. Second, we need to make sure that the ranges of f and g are subsets of $[0, 1]$. To do this, we use the *Fourier inversion formula*

$$f(u) = \sum_{\xi\in\widehat{G}} \hat{f}(\xi)\,\mathbf{e}(\xi\cdot u), u \in G$$

where we again use $\mathbf{e}(x) = e^{2\pi i x}$. By the triangle inequality, we have

$$|f(u) - 1/3| \leq \sum_{t \in G \setminus \{0\}} |\hat{f}(t)|. \tag{3}$$

By the same calculation, we also have

$$|g(u) - 1/3| \leq \sum_{t \in G \setminus \{0\}} |\hat{g}(t)|. \tag{4}$$

With these observations, we now construct f and g explicitly as follows.

We define f by setting

$$\hat{f}(s) = -\frac{2}{p^2}, \quad s \in G \setminus \{0\}$$

and g by setting

$$\hat{g}(s) = \frac{1}{p^2}, \quad s \in G \setminus \{0\}$$

We can take a prime p such that $p > |a_i|$ and $\gcd(a_i, p) = 1$ for each $i = 1, \ldots, 2m+1$, with

$$0 \leq \frac{1}{3} - \frac{2(p-1)}{p^2} = \frac{1}{3} - \sum_{t \in G \setminus \{0\}} |\hat{f}(t)| \leq f(u)$$
$$\leq \frac{1}{3} + \frac{2(p-1)}{p^2} = \frac{1}{3} + \sum_{t \in G \setminus \{0\}} |\hat{f}(t)| \leq 1$$

and

$$0 \leq \frac{1}{3} - \frac{(p-1)}{p^2} = \frac{1}{3} - \sum_{t \in G \setminus \{0\}} |\hat{g}(t)| \leq g(u)$$
$$\leq \frac{1}{3} + \frac{(p-1)}{p^2} = \frac{1}{3} + \sum_{t \in G \setminus \{0\}} |\hat{g}(t)| \leq 1.$$

for any $u \in G$. The above inequalities follow from Inequalities (3) and (4) and the specific values of $\hat{f}$ and $\hat{g}$. These ensure that the images of f and g lie in $[0, 1]$. We also remark that $\hat{f}$ and $\hat{g}$ are Hermitian since $\hat{f}$ and $\hat{g}$ are real-valued and both take only one value except at zero.

We also require that $0 < f(u) + g(u) < 1$ for any $u \in G$, which can be guaranteed by taking p sufficiently large such that

$$0 < \frac{2}{3} - \frac{3(p-1)}{p^2} < \frac{2}{3} + \frac{3(p-1)}{p^2} < 1.$$

Recalling the Fourier coefficients of f and g again, we find that the deviation is

$$-\left(\frac{2}{p^2}\right)^{2m+1}+\left(\frac{1}{p^2}\right)^{2m+1}+\left(\frac{1}{p^2}\right)^{2m+1}<0. \tag{5}$$

Therefore, every linear equation with $2m+1$ terms is 3-uncommon.

To show that those equations are also r-uncommon for $r>3$, we need to construct $r-1$ functions $f_1,\dots,f_{r-1}$ such that the deviation is negative. We generalize the construction used for the case $r=3$. As in the case when $r=3$, for $r>3$ we require that $\hat{f}_j(0)=\frac{1}{r}$ for every $j\in\{1,\dots,r-1\}$. Now we assign the values of their Fourier coefficients as follows. For $j=1,2$, we have

$$\hat{f}_1(s)=-\frac{2}{p^2},\quad s\in G\backslash\{0\},$$
$$\hat{f}_2(s)=\frac{1}{p^2},\quad s\in G\backslash\{0\}.$$

where p is a sufficiently large prime number in terms of $a_1,\dots,a_{2m+1}$, and also in terms of r so that $0\le f_j(u)\le 1, u\in G, j=1,2$.

Then we define $\hat{f}_j(s)=0$ for any $j=3,\dots,r-1$ and any $s\in G\backslash\{0\}$. Clearly, we have that each $\hat{f}_j$ is Hermitian and $f_j(u)\in[0,1]$ for any $u\in G$ and any $j=1,\dots,r-1$. By performing similar calculations to the case where $r=3$, we also get that the deviation is negative, and we can choose p such that $0<f_1(u)+\cdots+f_{r-1}(u)<1$ for any $u\in G$. This concludes the proof of Theorem 1. □

Remark 1. We note that if we choose a sufficiently large p, although the deviation in (5) is negative, it becomes negligible and the quantity is much smaller compared to the size of the group G. It would be interesting to obtain such an r-uncommon coloring explicitly.

To prove Theorem 2, we use the probabilistic method instead of using the Fourier method.

Proof of Theorem 2. We claim that if an equation is $(r-1)$-uncommon over the integers, then it is also r-uncommon over the integers for any $r\ge 3$. This clearly implies the statement of the theorem.

Note that the total number of solutions with $x_i=x_j$ for some $i\ne j$ is $O(n^{k-1})$. Thus, for large n, such solutions are negligible. Thus, we can focus on solutions with pairwise distinct coordinates.

Now, let $(x_1,\dots,x_k)$ be a solution of our linear equation in $[n]^k$ with $x_i\ne x_j$ for $i\ne j$, and let n be a sufficiently large integer such that the equation E is 2-uncommon over $[n]$. We consider an $(r-1)$-coloring of $[n]$ under which the equation is $(r-1)$-uncommon. We choose $\lfloor\frac{n}{r}\rfloor$ elements of $[n]$ uniformly at random, then we color those elements with the r-th color. Then if $(x_1,\dots,x_k)$ is monochromatic under the original coloring and $x_i \neq x_j$ for $i \neq j$, the probability that it will be monochromatic in the same color is

$$\left(\frac{r-1}{r}\right)^k.$$

If $(x_1, \ldots, x_k)$ is an arbitrary solution, it becomes monochromatic in the r-th color with probability

$$\left(\frac{1}{r}\right)^k.$$

Since the equation is $(r-1)$-uncommon, the expected proportion of monochromatic solutions under r colors is less than

$$\left(\frac{r-1}{r}\right)^k \frac{1}{(r-1)^{k-1}} + \left(\frac{1}{r}\right)^k = \left(\frac{1}{r}\right)^{k-1}.$$

Hence, there is an r-coloring of $[n]$ such that the equation E is r-uncommon.

Since we started with an equation that is $(r-1)$-uncommon, we get the result. $\square$

4. Open Problem

Costello and Elvin [1] used the Fourier method to show that equations of the form

$$x_1 + \cdots + x_m = x_{m+1} + \cdots + x_{2m} \tag{6}$$

are 2-common over the integers for any $m \geq 2$. While we proved that 2-uncommonness implies r-uncommonness for all $r \geq 3$, it is unknown whether 2-commonness implies r-commonness for $r \geq 3$. Even in the simplest case where $m = 2$ in (6), we are not yet able to determine whether it is 3-common or not.

This motivates the following question.

Question 1. Is there a linear equation that is r-uncommon for some $r \geq 3$ but s-common for some $s < r$?

We are particularly interested in equations with canceling partitions that are 2-common over the integers, since these remain poorly understood.

Acknowledgement. The author would like to thank Fernando Xuancheng Shao for helpful discussions and suggestions during the preparation of this work. The author is also indebted to anonymous referee for helpful suggestions and corrections.

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