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PRIME FACTORS WITH GIVEN MULTIPLICITY IN h-FREE AND h-FULL POLYNOMIALS OVER FUNCTION FIELDS

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Abstract

Let $f \in \mathbb{F}_q[T]$ be a monic polynomial over the finite field $\mathbb{F}_q$ of q elements and let $k \geq 1$ be a natural number. Following the work of Das, Elma, Kuo, and Liu, let $\omega_k(f)$ be the number of distinct monic irreducible factors of f with multiplicity k. We study the distribution of ω_k in $\mathbb{F}_q[T]$ when restricted to h-free polynomials and h-full polynomials. We show that a generalization of the Erdős–Kac Theorem restricted to the h-free polynomials is true for ω_1, but not for ω_k for $2 \leq k < h$, and similarly, a generalization of the Erdős–Kac Theorem restricted to the h-full polynomials is true for ω_h, but not for ω_k for $k \geq h+1$.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements, where q is a prime power. For $f \in \mathbb{F}_q[T]$, let

$$f = \alpha P_1^{v_1} \cdots P_r^{v_r} \tag{1}$$

be its prime factorization, where $P_j \in \mathbb{F}_q[T]$ is monic irreducible, $v_j \geq 1$, and $\alpha \in \mathbb{F}_q^*$.

Let $h \geq 2$ be a natural number. We say that $f \in \mathbb{F}_q[T]$ is *h-free* if $v_j < h$ for $j = 1, \ldots, r$ in Equation (1). Analogously, we say that f is *h-full* if $v_j \geq h$ for $j = 1, \ldots, r$ in Equation (1). Notice that for $h = 2$, we obtain the square-free and the square-full polynomials respectively. We denote by $\mathcal{S}_h$ and by $\mathcal{N}_h$ the sets of h-free and h-full polynomials respectively.

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The number of distinct prime divisors is given by $\omega(f) := r$. The distribution of values of ω has been extensively studied over the natural numbers. In 1940, Erdős and Kac famously proved that, for n a natural number, $\omega(n)$ has a limiting normal distribution in the sense that

$$\lim_{x\to\infty}\frac{1}{x}\left|\left\{3\le n\le x\,:\,\alpha\le\frac{\omega(n)-\log\log(n)}{\sqrt{\log\log(n)}}\le\beta\right\}\right|=\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt.$$

Various approaches to the Erdős and Kac Theorem have been pursued; see for example the works of Delange [5, 6], Halberstam [9], Billingsley [2], and Granville and Soundararajan [8].

The function field version of the Erdős–Kac Theorem was settled by W.-B. Zhang [18], namely, if we let $\mathcal{M}_n$ denote the monic polynomials of $\mathbb{F}_q[T]$ of degree n, then

$$\lim_{m\to\infty}\frac{\left|\left\{f\in\mathcal{M}_n\,:\,n\le m,\alpha\le\frac{\omega(f)-\log(n)}{\sqrt{\log(n)}}\le\beta\right\}\right|}{|\{f\in\mathcal{M}_n\,:\,n\le m\}|}=\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt.$$

This was then generalized by Liu [12, 13]. Rhoades [15] gave another proof, extending the methods of Granville and Soundararajan [8] to the function field setting.

It is also interesting to pose the question of whether the Erdős–Kac Theorem extends to subfamilies. Lalín and X. Zhang [11] proved the corresponding generalizations to the Erdős–Kac Theorem for h-free and h-full monic polynomials over $\mathbb{F}_q[T]$, by the method of moments. It is remarkable that, while a positive proportion of monic polynomials of a certain degree are h-free, and this proportion remains positive as the degree goes to infinity, the same is not true for h-full polynomials, which constitute a thin family as the degree goes to infinity. Therefore, the analogue to the Erdős–Kac Theorem is more surprising in this context.

In [7], Elma and Liu considered a refinement of $\omega(n)$ as follows. For $k\ge 1$ a natural number, let $\omega_k(n)$ denote the number of distinct prime factors of n with multiplicity k. They computed the first and second moments of $\omega_k(n)$, proved the analogue of the Erdős–Kac Theorem for $\omega_1(n)$, and showed that $\omega_k(n)$ for $k\ge 2$ does not have normal order $F(n)$ for any nondecreasing nonnegative function $F(n)$. This work was subsequently extended by Das, Elma, Kuo, and Liu [3] to the function field setting.

The goal of this manuscript is to explore the refined functions $\omega_k(f)$ in the case of the h-free and h-full families of monic polynomials in $\mathbb{F}_q[T]$. As one may expect from the above discussion, we obtain that $\omega_1(f)$ satisfies a limiting normal distribution over the h-free polynomials, while $\omega_k(f)$ for $2\le k<h$ does not. For the h-full polynomials, we have that $\omega_h(f)$ satisfies a limiting normal distribution, while $\omega_k(f)$ for $k\ge h+1$ does not. (In the h-free case, $\omega_k(f)$ is trivial for $k\ge h$ while in the h-full case, $\omega_k(f)$ is trivial for $k<h$.)

We adopt the following notation. Let $\mathcal{M}$ and $\mathcal{M}_n$ denote the sets of monic polynomials of $\mathbb{F}_q[T]$ and monic polynomials of degree n respectively. We let $\sum_P$

and $\prod_P$ denote the sum and product over all monic irreducible polynomials in $\mathbb{F}_q[T]$. For a polynomial $F(T) \in \mathbb{F}_q[T]$, the quantity $|F(T)| := q^{\deg(F)}$ denotes its norm or absolute value, and we set $|0| := 0$. We denote by $\zeta_q(s)$ the zeta function associated to $\mathbb{F}_q[T]$, defined precisely in Equation (10).

Define

$$B_1 = \gamma + \sum_P \left(\log\left(1 - \frac{1}{|P|}\right) + \frac{1}{|P|} \right)$$

to be the function field analogue of the first Mertens constant, where

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) \approx 0.57721566\dots$$

is the Euler–Mascheroni constant. The Mertens constant B_1 appears in estimates for $\sum_{n\le x} \omega(n)$. See [10, Theorem 430].

For $f \in \mathcal{M}$ and $P \in \mathcal{P}$, let $\nu_P(f)$ be the multiplicity of P in the factorization of f, that is, $\nu_P(f)$ is the integer such that $P^{\nu_P(f)} \mid f$ but $P^{\nu_P(f)+1} \nmid f$ (this includes the possibility of $\nu_P(f) = 0$ when $P \nmid f$). We have that

$$\omega(f) = \sum_{P|f} 1.$$

For a natural number $k \geq 1$, we define

$$\omega_k(f) := \sum_{\substack{P|f \\ \nu_P(f)=k}} 1.$$

Clearly we have

$$\omega(f) = \sum_{k\geq 1} \omega_k(f).$$

In [3, Theorem 1.1], Das, Elma, Kuo, and Liu prove that, as $n \to \infty$,

$$\frac{1}{|\mathcal{M}_n|} \sum_{f\in\mathcal{M}_n} \omega_1(f) = \log(n) + \left(B_1 - \sum_P \frac{1}{|P|^2} \right) + O\left(\frac{1}{n}\right). \tag{2}$$

For $k \geq 2$, $\varepsilon \in (0, 1/2)$, and as $n \to \infty$, they prove

$$\frac{1}{|\mathcal{M}_n|} \sum_{f\in\mathcal{M}_n} \omega_k(f) = \left(\sum_P \frac{1}{|P|^k} - \frac{1}{|P|^{k+1}} \right) + O_\varepsilon\left(q^{\frac{n}{k} - n + \varepsilon n}\right). \tag{3}$$

The authors of [3] compute the second moments in Theorem 1.2, obtaining results of the form

$$\frac{1}{|\mathcal{M}_n|} \sum_{f\in\mathcal{M}_n} \omega_1(f)^2 = \log^2(n) + C_2 \log(n) + C_3 + O\left(\frac{\log(n)}{n}\right),$$

and

$$\frac{1}{|\mathcal{M}_n|}\sum_{f\in\mathcal{M}_n}\omega_k(f)^2 = C_k' + O_\varepsilon\left(q^{\frac{n}{k}-n+\varepsilon n}\right),$$

where C_2, C_3, C_k' are certain precisely given constants (depending only on q).

In contrast, [11, Theorem 4.1] states that the moments over the h-free family are given, as $n\to\infty$, by

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega(f) = \log(n) + B_1 - \sum_P \frac{|P|-1}{|P|(|P|^h-1)} + O_\varepsilon\left(n^{\varepsilon-1}\right),$$

while

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega(f)^2 = \log^2(n) + S_2(h)\log(n) + S_3(h) + O_\varepsilon\left(n^{\varepsilon-1}\right),$$

for certain precise constants $S_2(h)$ and $S_3(h)$ depending on q and h.

Similarly, [11, Theorem 6.1] gives that

$$\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega(f) = \log\left(\frac{n}{h}\right) + N_1(h) + O_\varepsilon\left(n^{\varepsilon-1}\right),$$

while

$$\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega(f)^2 = \log^2\left(\frac{n}{h}\right) + N_2(h)\log\left(\frac{n}{h}\right) + N_3(h) + O_\varepsilon\left(n^{\varepsilon-1}\right),$$

and $N_1(h), N_2(h), N_3(h)$ are certain precisely given constants depending on q and h. In particular, setting $h=1$ for the h-full polynomials gives the moments of ω for the whole set of monic polynomials, namely,

$$\frac{1}{|\mathcal{M}_n|}\sum_{f\in\mathcal{M}_n}\omega(f) = \log(n) + N_1(1) + O_\varepsilon\left(n^{\varepsilon-1}\right),$$

and

$$\frac{1}{|\mathcal{M}_n|}\sum_{f\in\mathcal{M}_n}\omega(f)^2 = \log^2(n) + N_2(1)\log(n) + N_3(1) + O_\varepsilon\left(n^{\varepsilon-1}\right).$$

We remark that for $n\geq h$,

$$|\mathcal{S}_h\cap\mathcal{M}_n| = \frac{q^n}{\zeta_q(h)}$$

and

$$|\mathcal{N}_h\cap\mathcal{M}_n| = \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right) + O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right),$$

where ξ_h denotes a primitive complex root of unity of order h.

Throughout this article, the error terms have implied constants depending on q and h. This does not pose a problem, as we assume q and h to be fixed.

As previously stated, [3, Theorem 1.5] asserts that $\frac{\omega_1(f)-\log(n)}{\sqrt{\log(n)}}$ has a limiting normal distribution; [11, Theorem 4.2] gives the analogous result for $\frac{\omega(f)-\log(n)}{\sqrt{\log(n)}}$ restricted to $\mathcal{S}_h \cap \mathcal{M}_n$, while [11, Theorem 6.2] gives this for $\frac{\omega(f)-\log\left(\frac{n}{h}\right)}{\sqrt{\log\left(\frac{n}{h}\right)}}$ restricted to $\mathcal{N}_h \cap \mathcal{M}_n$.

In this work, we compute the first and second moments of ω_1 for h-free polynomials.

Theorem 1. *For $\varepsilon > 0$ and as $n \to \infty$, the first moment of ω_1 over the h-free polynomials of degree n is given by:*

$$\frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_1(f) = \log(n) + C_1(\mathcal{S}_h, 1) + O\left(\frac{1}{n}\right), \tag{4}$$

where

$$C_1(\mathcal{S}_h, 1) = B_1 - \sum_P \frac{|P|^{h-1} - 1}{|P|(|P|^h - 1)}.$$

The second moment is given by:

$$\begin{aligned} &\frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_1(f)^2 \\ &\qquad = \log^2(n) + C_2(\mathcal{S}_h, 1)\log(n) + C_3(\mathcal{S}_h, 1) + O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right), \end{aligned} \tag{5}$$

where

$$C_2(\mathcal{S}_h, 1) = \left(2\left(B_1 - \sum_P \frac{|P|^{h-1} - 1}{|P|(|P|^h - 1)}\right) + 1\right)$$

and

$$\begin{aligned} C_3(\mathcal{S}_h, 1) =& B_1^2 + B_1 - \zeta(2) - \sum_P \left(\frac{|P|^{h-2}(|P| - 1)}{|P|^h - 1}\right)^2 + \left(\sum_P \frac{|P|^{h-1} - 1}{|P|(|P|^h - 1)}\right)^2 \\ & - (2B_1 + 1)\sum_P \frac{|P|^{h-1} - 1}{|P|(|P|^h - 1)}. \end{aligned}$$

Finally, the variance is given by:

$$\begin{aligned} \mathrm{Var}_{h-\text{free},n}(\omega_1) :=& \frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_1(f)^2 - \left(\frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_1(f)\right)^2 \\ =& \log(n) + C_4(\mathcal{S}_h, 1) + O_\varepsilon\left(n^{\varepsilon - 1}\right), \end{aligned}$$

where

$$C_4(\mathcal{S}_h,1)=B_1-\zeta(2)-\sum_P\left(\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)^2-\sum_P\frac{|P|^{h-1}-1}{|P|(|P|^h-1)}.$$

We remark that the error term in the first moment is of better quality than the error term in the second moment, as this latter one contains an extra factor of n^ε. This is due to the fact that the first moment can be computed in two ways: one that uses the generating function for ω_1 directly, and another way, that deduces the moments from Equations (2) and (3) (obtained by Das, Elma, Kuo, and Liu with other methods), resulting in an improvement of the error term.

Another interesting observation is that for the case $h=2$, we have $\omega_1=\omega$, and the results for the moments coincide with the results from [11].

The techniques used to compute the first and second moments in the case of ω_1 can be pushed further to prove an analogue of the Erdős–Kac Theorem.

Theorem 2. *As $n\to\infty$, $\omega_1(f)$ with $f\in\mathcal{S}_h\cap\mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha\le\beta$,*

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\left|\left\{f\in\mathcal{S}_h\cap\mathcal{M}_n\ :\ \alpha\le\frac{\omega_1(f)-\log(n)}{\sqrt{\log(n)}}\le\beta\right\}\right|\to\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt.$$

Similarly, we obtain the first and second moments of ω_k for $1<k<h$ for h-free polynomials.

Theorem 3. *For $\varepsilon>0$ and as $n\to\infty$, the first moment of ω_k over the h-free polynomials of degree n is given by:*

$$\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_k(f)=\sum_P\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}+O_\varepsilon\left(q^{\frac{n}{k}-n+\varepsilon n}\right)\tag{6}$$

and the second moment is given by:

$$\begin{aligned}\frac{1}{|\mathcal{S}_h\cap\mathcal{M}_n|}&\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_k(f)^2\\&=\left(\sum_P\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\right)^2-\sum_P\left(\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\right)^2\\&\quad+\sum_P\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}+O_\varepsilon\left(q^{\frac{n}{k}-n+\varepsilon n}\right).\end{aligned}\tag{7}$$

Let $\mathcal{U}\subseteq\mathcal{M}$ and $g,G:\mathcal{U}\to\mathbb{R}_{\ge0}$ be two functions. We say that G is nondecreasing if $G(f_1)\ge G(f_2)$ for all f_1,f_2 with $\deg(f_1)\ge\deg(f_2)$. Then g is said to have normal order G (for a nondecreasing function G over $\mathcal{S}$) if for any $\varepsilon > 0$ the number of polynomials f with degree n that do not satisfy the inequalities

$$(1-\varepsilon)G(f) < g(f) < (1+\varepsilon)G(f)$$

is $o(\mathcal{U} \cap \mathcal{M}_n)$ as $n \to \infty$.

Theorems 1 and 2 imply in particular that $\omega_1(f)$ has normal order $\log(\deg(f))$ over the h-free polynomials. It is natural to pose the same question for ω_k when $k > 1$. We have the following negative result, which is consistent with the findings of [3] for the whole set of monic polynomials.

Theorem 4. *For $1 < k < h$, the function $\omega_k(f)$ does not have normal order $G(f)$ for any nondecreasing function $G : \mathcal{S}_h \cap \mathcal{M} \to \mathbb{R}_{\geq 0}$.*

Using the same methods, we can prove analogous results for $\mathcal{N}_h$. Before stating our results, we establish some notation. Let

$$K(P,h,j) := \frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1},$$

and recall that as $n \to \infty$,

$$\begin{aligned}|\mathcal{N}_h \cap \mathcal{M}_n| =& \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\\ &+ O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right).\end{aligned}$$

Theorem 5. *For $\varepsilon > 0$ and as $n \to \infty$, the first moment of ω_h over the h-full polynomials of degree n is given by:*

$$\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|}\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n}\omega_h(f) = \log\left(\frac{n}{h}\right) + C_1(\mathcal{N}_h, h) + O_\varepsilon\left(n^{\varepsilon-1}\right), \tag{8}$$

where

$$\begin{aligned}C_1(\mathcal{N}_h,h) =& B_1 + \frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|}\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\\ &\times\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right).\end{aligned}$$

The second moment is given by:

$$\begin{aligned}&\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|}\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n}\omega_h(f)^2\\ &\qquad = \log^2\left(\frac{n}{h}\right) + C_2(\mathcal{N}_h,h)\log\left(\frac{n}{h}\right) + C_3(\mathcal{N}_h,h) + O_\varepsilon\left(n^{\varepsilon-1}\right),\end{aligned} \tag{9}$$

where

$$C_2(\mathcal{N}_h,h) = 2B_1+1+\frac{2}{|\mathcal{N}_h\cap\mathcal{M}_n|}\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\times\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)$$

and

$$\begin{aligned}C_3(\mathcal{N}_h,h) =& B_1^2+B_1-\zeta(2)-\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\\&\times\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\left[\sum_P K(P,h,j)^2\right.\\&\left.+\left(\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)\right)^2+(2B_1+1)\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)\right].\end{aligned}$$

Finally, the variance is given by:

$$\begin{aligned}&\mathrm{Var}_{h-\mathrm{full},n}(\omega_h)\\&:=\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega_1(f)^2-\left(\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega_1(f)\right)^2\\&=\log\left(\frac{n}{h}\right)+B_1-\zeta(2)\\&\quad+\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\\&\quad\times\left[-\sum_P K(P,h,j)^2+\left(\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)\right)^2+\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)\right]\\&\quad-\left[\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\right.\\&\quad\left.\times\sum_P\left(K(P,h,j)-\frac{1}{|P|}\right)\right]^2+O_\varepsilon\left(n^{\varepsilon-1}\right).\end{aligned}$$

The techniques employed in the proof of the above statement can be pushed further to prove an analogue of the Erdős–Kac result.

Theorem 6. *As $n \to \infty$, $\omega_h(f)$ with $f \in \mathcal{N}_h \cap \mathcal{M}_n$ approaches a normal distribution, namely, for $\alpha \leq \beta$,*

$$\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \left| \left\{ f \in \mathcal{N}_h \cap \mathcal{M}_n \ : \ \alpha \leq \frac{\omega_h(f) - \log\left(\frac{n}{h}\right)}{\sqrt{\log\left(\frac{n}{h}\right)}} \leq \beta \right\} \right| \to \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt.$$

Similarly, we obtain the first and second moments of ω_k for $h < k$ for h-full polynomials.

Theorem 7. *For $\varepsilon > 0$ and as $n \to \infty$, the first moment of ω_k over the h-full polynomials of degree n is given by:*

$$\begin{aligned}
&\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \omega_k(f) \\
&\qquad = \frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \prod_P \left(1 - \frac{1}{|P|}\right) \left(1 + \frac{1}{|P|(1 - (q^{\frac{1}{h}} \xi_h^j)^{-\deg(P)})}\right) \\
&\qquad\quad \times \sum_P |P| (q^{\frac{1}{h}} \xi_h^j)^{-k \deg(P)} K(P, h, j) + O_\varepsilon\left(q^{-\frac{n}{h(h+1)} + \varepsilon n}\right)
\end{aligned}$$

and the second moment is given by:

$$\begin{aligned}
&\frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \omega_k(f)^2 \\
&\qquad = \frac{1}{|\mathcal{N}_h \cap \mathcal{M}_n|} \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \prod_P \left(1 - \frac{1}{|P|}\right) \left(1 + \frac{1}{|P|(1 - (q^{\frac{1}{h}} \xi_h^j)^{-\deg(P)})}\right) \\
&\qquad\quad \times \left[\sum_P \left(|P| (q^{\frac{1}{h}} \xi_h^j)^{-k \deg(P)} K(P, h, j) \right)^2 \right. \\
&\qquad\quad - \left(\sum_P |P| (q^{\frac{1}{h}} \xi_h^j)^{-k \deg(P)} K(P, h, j) \right)^2 \\
&\qquad\quad \left. + \sum_P |P| (q^{\frac{1}{h}} \xi_h^j)^{-k \deg(P)} K(P, h, j) \right] + O_\varepsilon\left(q^{-\frac{n}{h(h+1)} + \varepsilon n}\right).
\end{aligned}$$

Theorems 5 and 6 imply in particular that $\omega_h(f)$ has normal order $\log(\deg(f))$ over the h-full polynomials. It is natural to pose the same question for ω_k when $k > h$. We have the following negative result.

Theorem 8. *For $h < k$, the function $\omega_k(f)$ does not have normal order $G(f)$ for any nondecreasing function $G : \mathcal{N}_h \cap \mathcal{M} \to \mathbb{R}_{\geq 0}$.*

This article is organized as follows. We start by including some notation and preliminary statements in Section 2. The moments of ω_k over the h-free polynomials are treated in Section 3, including the first and second moment formulas, the normality results, and the analogue of the Erdős–Kac Theorem. Then Section 4 is organized likewise, for the h-full polynomials. Finally, we conclude with a discussion of possible directions of future research.

2. Notation and Preliminary Statements

Let $\mathcal{M}$ denote the set of monic polynomials over $\mathbb{F}_q[T]$ and let $\mathcal{M}_n$ (respectively $\mathcal{M}_{\leq n}$) denote the subset of $\mathcal{M}$ containing the polynomials of degree n (respectively degree $\leq n$). Let $\mathcal{P}$ denote the set of monic irreducible polynomials, and let $\mathcal{P}_n := \mathcal{P} \cap \mathcal{M}_n$ and $\mathcal{P}_{\leq n} := \mathcal{P} \cap \mathcal{M}_{\leq n}$ be defined analogously to the corresponding subsets of $\mathcal{M}$. The zeta function of $\mathbb{F}_q[T]$ is given by

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1}, \tag{10}$$

where the sum and the product converge for $\mathrm{Re}(s) > 1$. By summing over the degree and then over $\mathcal{M}_n$, one can prove that

$$\zeta_q(s) = \frac{1}{1 - q^{1-s}},$$

which gives a meromorphic continuation for $\zeta_q(s)$ to the whole complex plane, with simple poles when $q^s = q$. It is often convenient to consider the change of variables $u = q^{-s}$, which gives

$$\mathcal{Z}_q(u) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \prod_P \left(1 - u^{\deg(P)}\right)^{-1},$$

now converging absolutely for $|u| < \frac{1}{q}$, and having a meromorphic continuation to the complex plane with a simple pole at $u = \frac{1}{q}$.

We recall Perron's formula over $\mathbb{F}_q[T]$, which will be used throughout this article. (See for example [14, Equation 4.4.15] for the classical statement, and [4, Lemma 2.2] for the function field version.)

Theorem 9 (Perron's Formula)**.** *If the generating series $\mathcal{A}(u) = \sum_{f \in \mathcal{M}} a(f) u^{\deg(f)}$ is absolutely convergent in $|u| \leq r < 1$, then*

$$\sum_{f \in \mathcal{M}_n} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n} \frac{du}{u},$$

where $\oint$ denotes the integral over the circle centered at the origin and oriented counterclockwise.

The following result is due to Warlimont [17] in the context of arithmetical semigroups. It was later rediscovered by Afshar and Porritt [1] in the function field setting, and intensively used in [11]. It extends the Selberg–Delange method to function fields and is crucial for obtaining the first and second moments for $\omega_1(f)$ in all cases.

Theorem 10 ([17, 1])**.** *Let $C(u,z) = \sum_{n\geq 0} C_z(n)u^n$ and $B(u,z) = \sum_{n\geq 0} B_z(n)u^n$ be power series with coefficients depending on z satisfying $C(u,z) = B(u,z)\mathcal{Z}_q(u)^z$. Suppose also that, uniformly for $|z| \leq A$,*

$$\sum_{n\geq 0} \frac{|B_z(n)|}{q^n} n^{2A+2} \ll_A 1.$$

Then, uniformly for $|z| \leq A$ and $n \geq 1$, we have

$$C_z(n) = q^n \frac{n^{z-1}}{\Gamma(z)} B\left(1/q, z\right) + O_A\left(q^n n^{\mathrm{Re}(z)-2}\right),$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

For the h-full polynomial case we will need the following extension proven in [11].

Theorem 11 ([11])**.** *Let $C(u,z) = \sum_{n\geq 0} C_z(n)u^n$ and $B(u,z) = \sum_{n\geq 0} B_z(n)u^n$ be power series with coefficients depending on z satisfying $C(u,z) = B(u,z)\mathcal{Z}_q(u^h)^z$, where h is a positive integer. Suppose also that, uniformly for $|z| \leq A$,*

$$\sum_{n\geq 0} \frac{|B_z(n)|}{q^{\frac{n}{h}}} n^{2A+2} \ll_A 1.$$

Then, uniformly for $|z| \leq A$ and $n \geq 1$, we have

$$C_z(n) = \frac{q^{\frac{n}{h}} n^{z-1}}{h^z\Gamma(z)} \sum_{j=0}^{h-1} \xi_h^{jn} B\left((q^{\frac{1}{h}}\xi_h^j)^{-1}, z\right) + O_A\left(q^{\frac{n}{h}} n^{\mathrm{Re}(z)-2}\right),$$

where ξ_h denotes a primitive h-root of unity in $\mathbb{C}$.

To close this section, we recall a version of [11, Lemma 3.2] and take the opportunity to correct the statement and the proof, which are incorrect in [11]. This result will allow us to differentiate inside an error term after applying Theorems 10 and 11.

Lemma 1. *Let $G_n(z), f(z)$ be analytic functions at $z = 1$ such that*

$$G_n(z) = f(z)n^{z-1} + O\left(n^{\mathrm{Re}(z)-2}\right)$$

in a neighborhood of $z = 1$. Then, for an arbitrary small $\varepsilon > 0$, we have

$$G'_n(1) = \frac{\partial}{\partial z}(f(z)n^{z-1})\Big|_{z=1} + O_\varepsilon\left(n^{-1+\varepsilon}\right), \tag{11}$$

$$G_n^{(k)}(1) = \frac{\partial^k}{\partial z^k}(f(z)n^{z-1})\Big|_{z=1} + O_\varepsilon\left(n^{-1+\varepsilon}\right). \tag{12}$$

Proof. We begin by writing

$$G_n(z) = f(z)n^{z-1} + R_n(z).$$

Fix $\delta > 0$ and assume that $|z-1| < \delta$. For any $K > 0$ there is an $M = M_K$ such that if $n \geq M$, we have for $|z-1| < \delta$,

$$|R_n(z)| \leq Kn^{\mathrm{Re}(z)-2}.$$

Note that $f(z)n^{z-1}$ and $R_n(z)$ are analytic at $z = 1$. Therefore by Cauchy's integral formula for an arbitrarily small $\varepsilon > 0$,

$$\begin{aligned} G'_n(z) &= \frac{1}{2\pi i}\int_{|\xi-1|=\varepsilon} \frac{G_n(z)}{(\xi-1)^2}d\xi \\ &= \frac{1}{2\pi i}\int_{|\xi-1|=\varepsilon} \frac{f(z)n^{z-1}}{(\xi-1)^2}d\xi + \frac{1}{2\pi i}\int_{|\xi-1|=\varepsilon} \frac{R_n(z)}{(\xi-1)^2}d\xi \\ &= \frac{\partial}{\partial z}(f(z)n^{z-1}) + \frac{1}{2\pi i}\int_{|\xi-1|=\varepsilon} \frac{R_n(z)}{(\xi-1)^2}d\xi. \end{aligned}$$

Now note that for $n \geq M$ and $\varepsilon < \delta$,

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_{|\xi-1|=\varepsilon} \frac{R_n(z)}{(\xi-1)^2}d\xi\right| &\leq \frac{1}{2\pi}\int_{|\xi-1|=\varepsilon} \frac{Kn^{\mathrm{Re}(z)-2}}{\varepsilon^2}|d\xi| \\ &\leq \frac{1}{2\pi}2\pi\cdot\varepsilon\cdot\frac{1}{\varepsilon^2}Kn^{1+\varepsilon-2} = O\left(\frac{n^{-1+\varepsilon}}{\varepsilon}\right), \end{aligned}$$

where the last inequality follows from $|\mathrm{Re}(\xi)| \leq |\xi| \leq 1 + |\xi - 1| \leq 1 + \varepsilon$. This concludes the proof of Equation (11).

The general formula

$$G_n^{(k)}(1) = \frac{\partial^k}{\partial z^k}(f(z)n^{z-1})\Big|_{z=1} + O\left(\frac{n^{-1+\varepsilon}}{\varepsilon^k}\right)$$

can be proven similarly. □

3. Moments for h-Free Polynomials

In this section we treat the case of h-free monic polynomials, where $h \geq 2$ and $h \geq k \geq 1$. To prove Theorems 1 and 3, we start by considering the following Euler product, which converges absolutely for $|u| < \frac{1}{q}$ and gives the generating Dirichlet series for the h-free polynomials:

$$\begin{aligned}\frac{\mathcal{Z}_q(u)}{\mathcal{Z}_q(u^h)} =& \prod_P \left(\frac{1-u^{h\deg(P)}}{1-u^{\deg(P)}}\right)\\ =& \prod_P \left(1+u^{\deg(P)}+\cdots+u^{(h-1)\deg(P)}\right) = \sum_{f\in\mathcal{S}_h\cap\mathcal{M}} u^{\deg(f)}.\end{aligned}$$

We introduce the coefficient $\omega_k(f)$ by writing it as the exponent of an extra variable, which we will later differentiate. This preserves the additive structure of $\omega_k(f)$. We have

$$\begin{aligned}\mathcal{C}_{k,\mathcal{S}_h}(u,z) :=& \sum_{f\in\mathcal{S}_h\cap\mathcal{M}} z^{\omega_k(f)}u^{\deg(f)}\\ =& \prod_P (1+u^{\deg(P)}+\cdots+zu^{k\deg(P)}+\cdots+u^{(h-1)\deg(P)})\\ =& \prod_P \left(\frac{1-u^{h\deg(P)}}{1-u^{\deg(P)}}+(z-1)u^{k\deg(P)}\right),\end{aligned}$$

which converges absolutely for $|u| < \frac{1}{q}$ and $|z| < A$ for any positive constant A.

The above generating series can be used in two ways to compute the moments of $\omega_k(f)$. Firstly, we can use techniques such as Perron's formula (Theorem 9) and Theorem 10 to directly estimate the coefficient of u^n and obtain the moments by differentiating and evaluating at $z = 1$. Secondly, we can use the generating series to relate $\omega_k(f)$ over the h-free monic polynomials to the results from [3] for the whole family of monic polynomials. Next, we explain this second approach in detail.

Any $f \in \mathcal{M}$ can always be written as $f(T) = m(T)n(T)^h$ in an unique way, where $m(T)$ is h-free. We define $\mathfrak{m}_h(f) := m(f)$. Our next goal is to study $\omega_k \circ \mathfrak{m}_h$ for $k = 1, \ldots, h-1$, that is to say, we want to find $\sum_{f\in\mathcal{M}_n} \omega_k(\mathfrak{m}(f))$ and $\sum_{f\in\mathcal{M}_n} \omega_k(\mathfrak{m}(f))^2$.

We have the following generating function for $\omega_k \circ \mathfrak{m}_h$:

$$\begin{aligned}\sum_{f\in\mathcal{M}} z^{\omega_k(\mathfrak{m}_h(f))}u^{\deg(f)} =& \prod_P \Big(1+\cdots+u^{(k-1)\deg(P)}+zu^{k\deg(P)}\\ &+u^{(k+1)\deg(P)}+\cdots+zu^{(h+k)\deg(P)}+\cdots\Big)\\ =& \prod_P \left(\frac{1}{1-u^{\deg(P)}}+\frac{(z-1)u^{k\deg(P)}}{1-u^{h\deg(P)}}\right)\\ =& \mathcal{Z}_q(u^h)\mathcal{C}_{k,\mathcal{S}_h}(u,z). \qquad (13)\end{aligned}$$

We immediately see that there must be a relation between $\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_k(f)^j$ and $\sum_{f\in\mathcal{M}_n}\omega_k(\mathfrak{m}_h(f))^j$. We will make this more precise after we do a careful study of the generating functions.

Now, notice also that for $f\in\mathcal{M}_n$,

$$\omega_k(\mathfrak{m}_h(f))=\omega_k(f)+\omega_{h+k}(f)+\cdots+\omega_{\left(\lfloor\frac{n}{h}\rfloor-1\right)h+k}(f). \tag{14}$$

Thus, if we can describe the relationship between the moments of $\omega_k(f)$ over $\mathcal{S}_h$ and the moments of $\omega_k(\mathfrak{m}_h(f))$ in more precise terms, we can use Equation (14) and deduce Theorems 1 and 3 from the results of Das, Elma, Kuo, and Liu's, Equations (2) and (3). While we have not found an efficient way of doing this for the second moment, we do have a way of doing this for the first moment, which in the case of $\omega_1(f)$, leads to an improvement to the error term compared to what is obtained by directly applying the generating function.

We analyze the first two moments in the next subsections, as we consider two natural cases, according to whether $k=1$ or $1<k<h$, separately.

3.1. First and Second Moments of ω_1 for h-Free Polynomials

The case of ω_1 corresponds to $k=1$. We proceed to extract the singularity at $u=\frac{1}{q}$ in the generating function by writing

$$\mathcal{B}_{1,\mathcal{S}_h}(u,z):=\mathcal{Z}_q(u)^{-z}\mathcal{C}_{1,\mathcal{S}_h}(u,z).$$

Our goal is to apply Theorem 10. In order to proceed, we verify that the hypotheses are satisfied.

Lemma 2. *Let $\mathcal{B}_{1,\mathcal{S}_h}(u,z)=\sum_{n\geq 0}\mathcal{B}_z(n)u^n$. For $|z|\leq A$, $n\geq 2$, and $\sigma>\frac{1}{2}$,*

$$\sum_{0\leq a\leq n}\frac{|\mathcal{B}_z(a)|}{q^{\sigma a}}\leq c_{A,\sigma,q},$$

where $c_{A,\sigma,q}$ is a constant depending on A, σ, and q.

Proof. The argument follows very similar steps to those in the proof of [1, Proposition 2.5] and [11, Lemmas 3.1 and 5.1]. Let $b_z(f)$ be the function defined on the powers of monic irreducible polynomials P by

$$\begin{aligned}1+\sum_{j\geq 1}b_z(P^j)u^j=&(1+zu+u^2+\cdots+u^{h-1})(1-u)^z\\ =&(1+zu+u^2+\cdots+u^{h-1})\sum_{k=0}^{\infty}\binom{z}{k}(-u)^k\\ =&1-\frac{1}{2}(z^2+z-2)u^2+\cdots\end{aligned} \tag{15}$$

and extended multiplicatively to all $f \in \mathcal{M}$.

Now $\mathcal{B}_{1,\mathcal{S}_h}(u,z) = \sum_{f\in\mathcal{M}} b_z(f)u^{\deg(f)}$, and therefore, $\mathcal{B}_z(n) = \sum_{f\in\mathcal{M}_n} b_z(f)$. As seen on the right-hand side of Equation (15), $b_z(P) = 0$. We also remark that $\mathcal{B}_{1,\mathcal{S}_h}(u,z)$ converges absolutely for $|u| < \frac{1}{\sqrt{q}}$ and $|z| \le A$. By Cauchy's integral formula over $|u| = q^{-\frac{1+\varepsilon}{2}}$, we obtain

$$b_z(P^j) = \frac{1}{2\pi i}\oint_{|u|=q^{-\frac{1+\varepsilon}{2}}} (1 + zu + u^2 + \cdots + u^{h-1})(1-u)^z \frac{du}{u^{j+1}}.$$

Thus,

$$|b_z(P^j)| \le q^{\frac{j}{2}(1+\varepsilon)} M_A$$

for $j \ge 2$, where

$$M_A := \sup_{|z|\le A, |u|\le\sqrt{\frac{2}{3}}} \left|(1 + zu + u^2 + \cdots + u^{h-1})(1-u)^z\right|$$

is a constant depending on A. The rest of the proof proceeds exactly as in [11, Lemma 3.1] to obtain

$$\sum_{0\le a\le n} \frac{|\mathcal{B}_z(a)|}{q^{\sigma a}} \ll_{\sigma.q} \exp\left(\frac{M_A}{q^{2\sigma-1}-1}\right),$$

where the implied constant is independent of n. □

Since $a^{2A+2} < q^{a/3}$ as a approaches infinity, it follows from Lemma 2 that

$$\sum_{a\ge 0} \frac{|\mathcal{B}_z(a)|}{q^a} a^{2A+2} < \sum_{a\ge 0} \frac{|\mathcal{B}_z(a)|}{q^{\frac{2a}{3}}} \ll_A 1$$

uniformly for $|z| \le A$. Thus we can apply Theorem 10 to $\mathcal{B}_{1,\mathcal{S}_h}(u,z)$.

Proof of Theorem 1. Recall that we have

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}} z^{\omega_1(f)} u^{\deg(f)} = \mathcal{C}_{1,\mathcal{S}_h}(u,z) = \mathcal{B}_{1,\mathcal{S}_h}(u,z)\mathcal{Z}_q(u)^z.$$

Applying Theorem 10, this gives

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} z^{\omega_1(f)} = q^n \frac{n^{z-1}}{\Gamma(z)} \mathcal{B}_{1,\mathcal{S}_h}(1/q,z) + O_A\left(q^n n^{\mathrm{Re}(z)-2}\right). \tag{16}$$

Differentiating both sides of Equation (16) with respect to z for z close to 1, and applying Equation (11), we have that

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \omega_1(f) z^{\omega_1(f)-1} = \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)}\right)' q^n n^{z-1} + \frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)} q^n \log(n) n^{z-1} + O_z(1) O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right). \tag{17}$$

Evaluating Equation (17) at $z = 1$, we have

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_1(f) = \frac{\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma(1) - \mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma'(1)}{\Gamma(1)^2}q^n + \frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)}{\Gamma(1)}q^n\log(n) + O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right). \tag{18}$$

Recall that $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$. Notice that the evaluation of $\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)$ is particularly simple:

$$\mathcal{B}_{1,\mathcal{S}_h}(1/q,1) = \prod_P\left(1 - \frac{1}{|P|^h}\right) = \frac{1}{\zeta_q(h)}.$$

In addition, the logarithmic derivative of $\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)$ gives

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)} = \sum_P\left(\log\left(1-\frac{1}{|P|}\right) + \frac{|P|^{h-2}(|P|-1)}{|P|^h - 1 + (z-1)|P|^{h-2}(|P|-1)}\right). \tag{19}$$

Applying the above identities to Equation (18), we obtain the proof of Equation (4) (with an error term of $O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right)$).

We now proceed to prove Equation (5). Multiplying Equation (17) by z, differentiating both sides with respect to z for z close to 1, and applying Equation (12), we obtain

$$\begin{aligned}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_1(f)^2 z^{\omega_1(f)-1} = &\left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)}\right)'' q^n z n^{z-1} \\ &+ \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)}\right)' q^n(n^{z-1} + 2zn^{z-1}\log(n)) \\ &+ \frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)}q^n\log(n)(n^{z-1} + zn^{z-1}\log(n)) \\ &+ O_z(1)O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned} \tag{20}$$

Now observe that

$$\begin{aligned}\left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)}{\Gamma(z)}\right)'' &= \left(\frac{\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)\Gamma(z) - \mathcal{B}_{1,\mathcal{S}_h}(1/q,z)\Gamma'(z)}{\Gamma(z)^2}\right)' \\ &= \frac{1}{\Gamma(z)^3}\left(\frac{\partial^2}{\partial z^2}\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)\Gamma(z)^2 - 2\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)\Gamma(z)\Gamma'(z)\right. \\ &\quad \left.+\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)(2\Gamma'(z)^2 - \Gamma(z)\Gamma''(z))\right).\end{aligned}$$

In addition, multiplying Equation (19) by $\mathcal{B}_{1,\mathcal{S}_h}(1/q,z)$, taking the derivative, and evaluating at $z = 1$, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1) =&\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)\\&-\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\sum_P\left(\frac{|P|^{h-2}(|P|-1)}{(|P|^h-1)}\right)^2\\=&\frac{1}{\zeta_q(h)}\left[\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)\right]^2\\&-\frac{1}{\zeta_q(h)}\sum_P\left(\frac{|P|^{h-2}(|P|-1)}{(|P|^h-1)}\right)^2. \qquad (21)\end{aligned}$$

Evaluating Equation (20) at $z=1$ gives

$$\begin{aligned}\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_1(f)^2 =&\frac{1}{\Gamma(1)^3}\left(\frac{\partial^2}{\partial z^2}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma(1)^2-2\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma(1)\Gamma'(1)\right.\\&\left.+\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)(2\Gamma'(1)^2-\Gamma(1)\Gamma''(1))\right)q^n\\&+\frac{\frac{\partial}{\partial z}\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma(1)-\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)\Gamma'(1)}{\Gamma(1)^2}q^n(1+2\log(n))\\&+\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q,1)}{\Gamma(1)}q^n\log(n)(1+\log(n))+O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned}$$

Inserting Equation (21) in the above equation, and using the fact that $\Gamma''(1)=\gamma^2+\zeta(2)$, we obtain

$$\begin{aligned}&\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n}\omega_1(f)^2\\&=\frac{q^n(\log(n))^2}{\zeta_q(h)}+\frac{q^n(\log(n))}{\zeta_q(h)}\left(2\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)\right.\\&\left.+2\gamma+1\right)+\frac{q^n}{\zeta_q(h)}\left[\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)\right]^2\\&-\frac{q^n}{\zeta_q(h)}\sum_P\left(\frac{|P|^{h-2}(|P|-1)}{(|P|^h-1)}\right)^2\\&+\frac{2\gamma+1}{\zeta_q(h)}q^n\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{|P|^{h-2}(|P|-1)}{|P|^h-1}\right)\\&+\frac{\gamma^2+\gamma-\zeta(2)}{\zeta_q(h)}q^n+O_\varepsilon\left(\frac{q^n}{n^{1-\varepsilon}}\right).\end{aligned}$$

Combining this with the definition of B_1, we obtain Equation (5).

The variance can be directly computed from Equations (4) and (5) by recalling that [11, Lemma 3.3] gives, for $n \geq h$,

$$|\mathcal{S}_h \cap \mathcal{M}_n| = \frac{q^n}{\zeta_q(h)}.$$

Now we consider the improvement on the error term of the first moment that can be obtained by working with Equations (13), (14), as well as Equations (2) and (3). From the above discussion of the generating function and Equation (13), we deduce that

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_k(f)^j = \frac{1}{\zeta_q(h)} \sum_{f \in \mathcal{M}_n} \omega_k(\mathfrak{m}_h(f))^j. \tag{22}$$

Now, combining the above with Equations (14), (2), and (3) gives

$$\begin{aligned}\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \omega_1(f) =& \frac{1}{\zeta_q(h)} \left[q^n \log(n) + q^n \left(B_1 - \sum_P \frac{1}{|P|^2} \right) \right. \\ & \left. + \sum_{j=1}^{\lfloor \frac{n}{h} \rfloor - 1} q^n \left(\sum_P \frac{1}{|P|^{jh+1}} - \frac{1}{|P|^{jh+2}} \right) \right] \\ & + O\left(\frac{q^n}{n} \right) + O_\varepsilon\left(n q^{\frac{n}{h} + \varepsilon n} \right) \\ =& \frac{q^n \log(n)}{\zeta_q(h)} + \frac{q^n}{\zeta_q(h)} \left(B_1 - \sum_P \frac{|P|^{h-1} - 1}{|P|(|P|^h - 1)} \right) + O\left(\frac{q^n}{n} \right),\end{aligned}$$

yielding Equation (4) with the better error term of size $\frac{q^n}{n}$.

This concludes the proof of the first and second moments of $\omega_1(f)$. □

3.2. First and Second Moments of ω_k for h-Free Polynomials

Here we consider the case $1 < k < h$. We have that $\mathcal{C}_{k,\mathcal{S}_h}(u, z)$ has a pole of order 1 at $u = \frac{1}{q}$. We extract it as follows:

$$\begin{aligned}\mathcal{B}_{k,\mathcal{S}_h}(u, z) =& \mathcal{Z}_q(u)^{-1} \mathcal{C}_{k,\mathcal{S}_h}(u, z) \\ =& \prod_P \left(1 - u^{h \deg(P)} + (z-1) u^{k \deg(P)} (1 - u^{\deg(P)}) \right),\end{aligned}$$

where $\mathcal{B}_{k,\mathcal{S}_h}(u, z)$ is absolutely convergent for $|u| < q^{-\frac{1}{k}}$ and $|z| \leq A$.

By Perron's formula (Theorem 9), we have that

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} z^{\omega_k(f)} = \frac{1}{2\pi i} \oint \frac{\mathcal{B}_{k,\mathcal{S}_h}(u, z)}{(1 - qu) u^n} \frac{du}{u},$$

where the integral takes place on a small circle around the origin. We move the circle to $|u| = q^{-\varepsilon-\frac{1}{k}}$ and obtain the residue at $u = \frac{1}{q}$. This gives

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} z^{\omega_k(f)} = -\operatorname{Res}_{u=\frac{1}{q}} \frac{\mathcal{B}_{k,\mathcal{S}_h}(u,z)}{(1-qu)u^{n+1}} + \frac{1}{2\pi i}\oint_{|u|=q^{-\varepsilon-\frac{1}{k}}} \frac{\mathcal{B}_{k,\mathcal{S}_h}(u,z)}{(1-qu)u^n}\frac{du}{u}$$
$$= \mathcal{B}_{k,\mathcal{S}_h}(1/q,z)q^n + O_z(1)O_\varepsilon\left(q^{\frac{n}{k}+\varepsilon n}\right). \tag{23}$$

Proof of Theorem 3. In order to recover the first moment, we differentiate and evaluate Equation (23) at $z = 1$:

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \omega_k(f) = \frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1)q^n + O_\varepsilon\left(q^{\frac{n}{k}+\varepsilon n}\right). \tag{24}$$

The logarithmic derivative of $\mathcal{B}_{k,\mathcal{S}_h}(1/q,z)$ gives

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{S}_h}(1/q,z)}{\mathcal{B}_{k,\mathcal{S}_h}(1/q,z)} = \sum_P \frac{|P|^{h-k-1}(|P|-1)}{|P|^h - 1 + (z-1)|P|^{h-k-1}(|P|-1)},$$

and thus

$$\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1) = \frac{1}{\zeta_q(h)}\sum_P \frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}. \tag{25}$$

Substituting the above in Equation (24) gives the first moment, Equation (6).

We proceed to compute the second moment. Differentiating Equation (23), multiplying by z, differentiating again, and setting $z = 1$, we have

$$\sum_{f\in\mathcal{S}_h\cap\mathcal{M}_n} \omega_k(f)^2 = \frac{\partial^2}{\partial z^2}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1)q^n + \frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1)q^n + O_\varepsilon\left(q^{\frac{n}{k}+\varepsilon n}\right). \tag{26}$$

For the second derivative we have

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1)\\
&= \frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{S}_h}(1/q,1)\sum_P \frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\\
&\quad - \mathcal{B}_{k,\mathcal{S}_h}(1/q,1)\sum_P\left(\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\right)^2\\
&= \frac{1}{\zeta_q(h)}\left[\left(\sum_P \frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\right)^2 - \sum_P\left(\frac{|P|^{h-k-1}(|P|-1)}{|P|^h-1}\right)^2\right].
\end{aligned}$$

Combining this with Equation (25) in Equation (26) gives Equation (7), the desired result.

As a final note, we remark that we could have computed the first moment from Equation (22) by combining with Equations (14) and (3). This alternative approach does not improve the error term in Equation (6). □

3.3. Normal Order and an Erdős–Kac Result for h-Free Polynomials

The goal of this section is to prove Theorem 2, namely the Erdős–Kac result for ω_1 over the h-free polynomials, and Theorem 4, which investigates the normal order of the functions ω_k over the h-free polynomials.

Proof of Theorem 2. Our argument follows very closely the proof of [11, Theorem 1.3]. In order to prove the statement we will show that as $n \to \infty$,

$$\frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \left(\frac{\omega_1(f) - \log(n)}{\sqrt{\log(n)}} \right)^v \to C_v$$

where

$$C_v = \begin{cases} \frac{v!}{2^{\frac{v}{2}} \left(\frac{v}{2}\right)!} & v \text{ even}, \\ 0 & v \text{ odd}. \end{cases} \tag{27}$$

Let us study the higher moments of ω_1. We consider again the moment generating function with $z = e^t$:

$$\mathcal{C}_{1,\mathcal{S}_h}(u, e^t) = \sum_{f \in \mathcal{S}_h \cap \mathcal{M}} e^{t\omega_1(f)} u^{\deg(f)} = \prod_P \left(1 + e^t u^{\deg(P)} + \cdots + u^{(h-1)\deg(P)}\right).$$

Evaluating at $z = e^t$ will allow us to differentiate multiple times in a single step and recover the moments via the generating function

$$\mathbb{E}\left(\omega_1^\ell\right) = \mathbb{E}\left(e^{t\omega_1}\right)^{(\ell)}\Big|_{t=0}. \tag{28}$$

We extract the singularity of $\mathcal{C}_{1,\mathcal{S}_h}(u, e^t)$ at $u = \frac{1}{q}$ as

$$\mathcal{B}_{1,\mathcal{S}_h}(u, e^t) = \mathcal{Z}_q(u)^{-e^t} \mathcal{C}_{1,\mathcal{S}_h}(u, e^t).$$

By applying Theorem 10 we obtain

$$\sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} e^{\omega_1(f)t} = q^n \frac{n^{e^t-1}}{\Gamma(e^t)} \mathcal{B}_{1,\mathcal{S}_h}\left(1/q, e^t\right) + O_A\left(q^n n^{\mathrm{Re}(e^t)-2}\right).$$

By Equation (28), we have

$$\mathbb{E}\left(\omega_1^\ell\right) = \zeta_q(h) \sum_{j=0}^{\ell} \binom{\ell}{j} (n^{e^t-1})^{(j)} \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}\left(1/q, e^t\right)}{\Gamma(e^t)} \right)^{(\ell-j)} \Bigg|_{t=0} + O_\varepsilon\left(\frac{1}{n^{1-\varepsilon}}\right). \tag{29}$$

Recall from Equation (24) in [11], that for $j \geq 1$,

$$(n^{e^t-1})^{(j)} = n^{e^t-1} \sum_{m=1}^{j} \left\{ {j \atop m} \right\} (e^t \log(n))^m = n^{e^t-1} T_j(e^t \log(n)), \tag{30}$$

where the $\left\{ {j \atop m} \right\}$ are the Stirling numbers of the second kind, and the T_j are the Touchard polynomials [16].

By combining Equation (30) with Equation (29), we have:

$$\mathbb{E}\left(\omega_1^\ell\right) = \zeta_q(h) \sum_{j=0}^{\ell} \binom{\ell}{j} T_j(\log(n)) \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q, e^t)}{\Gamma(e^t)} \right)^{(\ell-j)} \Bigg|_{t=0} + O_\varepsilon\left(\frac{1}{n^{1-\varepsilon}}\right). \quad (31)$$

Notice that

$$\begin{aligned} &\frac{1}{|\mathcal{S}_h \cap \mathcal{M}_n|} \sum_{f \in \mathcal{S}_h \cap \mathcal{M}_n} \left(\frac{\omega_1(f) - \log(n)}{\sqrt{\log(n)}} \right)^v \\ &= \frac{1}{(\log n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\omega_1^\ell)(-1)^{v-\ell} (\log n)^{v-\ell}. \end{aligned}$$

Combining the above with Equation (31), we then obtain

$$\begin{aligned} &\frac{1}{(\log n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\omega_1^\ell)(-1)^{v-\ell} (\log n)^{v-\ell} \\ &= \frac{\zeta_q(h)}{(\log n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} T_j(\log(n)) \\ &\times \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q, e^t)}{\Gamma(e^t)} \right)^{(\ell-j)} \Bigg|_{t=0} (-1)^{v-\ell} (\log n)^{v-\ell} + O_\varepsilon\left(\frac{1}{n^{1-\varepsilon}}\right). \quad (32) \end{aligned}$$

Consider the change of variables $u = v - \ell$, $m = v - \ell + j$. Then, the main term in Equation (32) becomes

$$\begin{aligned} &\frac{\zeta_q(h)}{(\log n)^{\frac{v}{2}}} \sum_{m=0}^{v} \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q, e^t)}{\Gamma(e^t)} \right)^{(v-m)} \Bigg|_{t=0} \\ &\times \sum_{u=0}^{m} \binom{v}{u} \binom{v-u}{m-u} T_{m-u}(\log n)(-1)^u (\log n)^u \\ &= \frac{\zeta_q(h)}{(\log n)^{\frac{v}{2}}} \sum_{m=0}^{v} \binom{v}{m} \left(\frac{\mathcal{B}_{1,\mathcal{S}_h}(1/q, e^t)}{\Gamma(e^t)} \right)^{(v-m)} \Bigg|_{t=0} \\ &\times \sum_{u=0}^{m} \binom{m}{u} T_{m-u}(\log n)(-1)^u (\log n)^u. \quad (33) \end{aligned}$$

Since the generating function for the Touchard polynomials [16] is

$$e^{x(e^t-1)} = \sum_{m=0}^{\infty} \frac{T_m(x)}{m!} t^m,$$

one can see that the generating function for the inner sum in Equation (33) is given by

$$e^{x(e^t-1-t)} = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{u=0}^{m} \binom{m}{u} T_{m-u}(x)(-1)^u x^u.$$

Notice that the coefficient of x^u in the power series of $e^{x(e^t-1-t)}$ is given by $\frac{(e^t-1-t)^u}{u!}$, whose lowest power of t is t^{2u}. Therefore, thinking of $e^{x(e^t-1-t)}$ as a power series in t, we have that the coefficient of t^v is a polynomial in x of degree at most $\lfloor \frac{v}{2} \rfloor$.

Now suppose that v is even. Then, the coefficient of $x^{\frac{v}{2}} t^v$ in $e^{x(e^t-1-t)}$ is given by $\frac{1}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}$. Back to the inner sum in Equation (33), this gives a leading coefficient of $\frac{m!}{2^{\frac{m}{2}}\left(\frac{m}{2}\right)!}$ for $(\log n)^{\frac{m}{2}}$ when m is even. Incorporating this information in Equation (33), we get, for v even,

$$\begin{aligned}
&\frac{1}{(\log n)^{\frac{v}{2}}} \sum_{\ell=0}^{v} \binom{v}{\ell} \mathbb{E}(\Omega(f)^\ell)(-1)^{v-\ell}(\log n)^{v-\ell} \\
&\qquad = \frac{\zeta_q(h)}{(\log n)^{\frac{v}{2}}} \frac{\mathcal{B}(1/q,1)}{\Gamma(1)} \sum_{u=0}^{v} \binom{v}{u} T_{v-u}(\log n)(-1)^u (\log n)^u + O\left(\frac{1}{\log n}\right) \\
&\qquad = \frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!} + O\left(\frac{1}{\log n}\right),
\end{aligned}$$

while for v odd we get

$$O\left(\frac{1}{\sqrt{\log n}}\right),$$

as desired. ∎

Before proceeding to the proof of Theorem 4, we need an auxiliary result.

Lemma 3. *Let $n \geq 0$, $\ell > 0$ be integers and let $P \in \mathcal{P}$. Then, as $n \to \infty$,*

$$\sum_{\substack{f \in \mathcal{S}_\ell \cap \mathcal{M}_n \\ (f,P)=1}} 1 = \frac{q^n}{\zeta_q(\ell)} \left(\frac{1-|P|^{-1}}{1-|P|^{-\ell}}\right) + O_{\ell,\deg(P)}(1).$$

Proof. We consider the generating function

$$\sum_{\substack{f \in \mathcal{S}_\ell \cap \mathcal{M} \\ (f,P)=1}} u^{\deg(P)} = \frac{\mathcal{Z}_q(u)(1-u^{\deg(f)})}{\mathcal{Z}_q(u^\ell)(1-u^{\ell \deg(P)})}.$$

We remark that the above generating function has simple poles at $u = \frac{1}{q}$ and $u = \xi^j_{\ell \deg(P)}$ for $j = 0, \ldots, \ell \deg(P) - 1$, and no other poles.

By Perron's formula (Theorem 9), and by moving the integral to $|u| = R$ with $R \to \infty$, we have

$$\begin{aligned}\sum_{\substack{f\in\mathcal{S}_\ell\cap\mathcal{M}_n\\(f,P)=1}} 1 =&\frac{1}{2\pi i}\oint \frac{(1-qu^\ell)(1-u^{\deg(P)})}{(1-qu)(1-u^{\ell\deg(P)})}\frac{du}{u^{n+1}}\\ =&-\operatorname{Res}_{u=\frac{1}{q}}\frac{(1-qu^\ell)(1-u^{\deg(P)})}{(1-qu)(1-u^{\ell\deg(P)})u^{n+1}}\\ &-\sum_{j=0}^{\ell\deg(P)-1}\operatorname{Res}_{u=\xi^j_{\ell\deg(P)}}\frac{(1-qu^\ell)(1-u^{\deg(P)})}{(1-qu)(1-u^{\ell\deg(P)})u^{n+1}}\\ =&\frac{q^n}{\zeta_q(\ell)}\left(\frac{1-|P|^{-1}}{1-|P|^{-\ell}}\right)+O_{\ell,\deg(P)}(1).\end{aligned}$$

□

Notice in particular that Lemma 3 implies that

$$\sum_{\substack{f\in\mathcal{S}_\ell\cap\mathcal{M}_n\\ \nu_P(f)=k}} 1 = \sum_{\substack{f\in\mathcal{S}_h\cap\mathcal{M}_{n-k\deg(P)}\\(f,P)=1}} 1 = \frac{q^{n-k\deg(P)}}{\zeta_q(\ell)}\left(\frac{1-|P|^{-1}}{1-|P|^{-\ell}}\right)+O_{h,\deg(P)}(1).$$

Proof of Theorem 4. We follow the argument given in [3, Theorem 1.4]. Let $G(f)$ be a nondecreasing function $G : \mathcal{S}_h \cap \mathcal{M} \to \mathbb{R}_{\geq 0}$. First assume that there is an $f_0 \in \mathcal{S}_h \cap \mathcal{M}$ such that $G(f_0) > 0$. Therefore, $G(f) > 0$ for all $f \in \mathcal{S}_h \cap \mathcal{M}$ such that $\deg(f) > \deg(f_0)$.

Let $n > \deg(f_0)$ and consider the following set:

$$\mathcal{O}_{\mathcal{S},0}(n,h) := \{f \in \mathcal{S}_h \cap \mathcal{M}_n \ : \ \omega_k(f) = 0\}.$$

It can be seen that

$$\mathcal{S}_k \cap \mathcal{M}_n = \mathcal{S}_h \cap \mathcal{M}_n \cap \mathcal{S}_k \subseteq \mathcal{O}_{\mathcal{S},0}(n,h),$$

and therefore

$$|\mathcal{O}_{\mathcal{S},0}(n,h)| \geq |\mathcal{S}_k \cap \mathcal{M}_n| = \frac{q^n}{\zeta_q(k)}$$

for $n \geq k$.

This means that $|\mathcal{O}_{\mathcal{S},0}(n,h)|$ is $\gg q^n$. In particular, the set of monic h-free polynomials f for which $G(f) > 0$ and $\omega_k(f) = 0$ is not $o(q^n)$. Clearly, for those f, we have

$$|\omega_k(f) - G(f)| > \frac{G(f)}{2}$$

is satisfied. That means that $\omega_k(f)$ does not have normal order G when G is not the constant function 0.

Now, if $G(f) = 0$ for all $f \in \mathcal{S}_h \cap \mathcal{M}$ we define:

$$\mathcal{O}_{\mathcal{S},1}(n,h) := \{f \in \mathcal{S}_h \cap \mathcal{M}_n \ : \ \omega_k(f) = 1\}.$$

Let $P \in \mathcal{P}_1$ be a fixed monic irreducible polynomial of degree 1. It can be seen that

$$\begin{aligned}|\mathcal{O}_{\mathcal{S},1}(n,h)| \geq &\sum_{\substack{f \in \mathcal{S}_h \cap \mathcal{M}_n,\, \nu_P(f)=k \\ \nu_Q(f)<k, \text{ for all } Q \in \mathcal{P},\, Q \neq P}} 1 \geq \sum_{\substack{f \in \mathcal{S}_k \cap \mathcal{M}_{n-k} \\ (f,P)=1}} 1 \\ =&\frac{q^{n-k}}{\zeta_q(k)} \left(\frac{1-q^{-1}}{1-q^{-k}} \right) + O_k(1),\end{aligned}$$

where we have used Lemma 3.

This means that $|\mathcal{O}_{\mathcal{S},1}(n,h)|$ is $\gg q^n$. In particular, the set of monic h-free polynomials f for which $G(f) = 0$ and $\omega_k(f) = 1$ is not $o(q^n)$. For those f, we have that

$$|\omega_k(f) - G(f)| > \frac{G(f)}{2}.$$

Therefore ω_k does not have normal order G when G is the constant function 0. □

4. Moments for h-Full Polynomials

In this section we prove the analogous results for the case of h-full polynomials, where $k \geq h \geq 2$. To prove Theorems 5 and 7 we consider the following Euler product, which converges absolutely for $u < q^{-\frac{1}{h}}$ and gives the generating Dirichlet series for the h-full polynomials:

$$\prod_P \left(1 + u^{h \deg(P)} + u^{(h+1)\deg(P)} + \cdots \right) = \sum_{f \in \mathcal{N}_h \cap \mathcal{M}} u^{\deg(f)}.$$

We introduce the coefficient $\omega_k(f)$ as in previous sections, the process being completely analogous.

$$\begin{aligned}\mathcal{C}_{k,\mathcal{N}_h}(u,z) := &\sum_{f \in \mathcal{N}_h \cap \mathcal{M}} z^{\omega_k(f)} u^{\deg(f)} \\ =&\prod_P \left(1 + u^{h \deg(P)} + \cdots + z u^{k \deg(P)} + \cdots \right) \\ =&\prod_P \left(\frac{1 - u^{\deg(P)} + u^{h \deg(P)}}{1 - u^{\deg(P)}} + (z-1) u^{k \deg(P)} \right),\end{aligned}$$

which converges absolutely for $|u| < q^{-\frac{1}{h}}$ and $|z| < A$ for any positive constant A.

For the h-full polynomials there are two cases, according to whether $k = h$ or $h < k$. These two cases will be considered separately.

4.1. First and Second Moments of ω_h for h-Full Polynomials

We consider here the case $k = h$. We proceed to extract the singularity at $u = q^{-\frac{1}{h}}$ by writing

$$\mathcal{B}_{h,\mathcal{N}_h}(u,z) := \mathcal{Z}_q(u^h)^{-z}\mathcal{C}_{h,\mathcal{N}_h}(u,z)$$

Our goal is to apply Theorem 11, for which we need to verify that the hypotheses are satisfied.

Lemma 4. *Let $\mathcal{B}_{h,\mathcal{N}_h}(u,z) = \sum_{n\geq 0}\mathcal{B}_z(n)u^n$. For $|z| \leq A$, $n \geq 2$ and $\sigma > \frac{1}{h+1}$,*

$$\sum_{0\leq a\leq n}\frac{|\mathcal{B}_z(a)|}{q^{\sigma a}} \leq c_{A,\sigma,q},$$

where $c_{A,\sigma}$ is a constant depending on A, σ, and q.

Proof. The argument follows very similar steps to those in the proof of [1, Proposition 2.5] and [11, Lemmas 3.1 and 5.1]. Let $b_z(f)$ be the function defined on the powers of monic irreducible polynomials P by

$$1 + \sum_{j\geq 1} b_z(P^j)u^j = (1 + zu^h + u^{h+1} + \cdots)(1-u^h)^z, \tag{34}$$

and extended multiplicatively to all $f \in \mathcal{M}$.

Now $\mathcal{B}_{h,\mathcal{N}_h}(u,z) = \sum_{f\in\mathcal{M}} b_z(f)u^{\deg(f)}$, and therefore, $\mathcal{B}_z(n) = \sum_{f\in\mathcal{M}_n} b_z(f)$. Expanding the right-hand side of Equation (34), we see that $b_z(P^j) = 0$ for $j \leq h$. We remark that $\mathcal{B}_{1,\mathcal{N}_h}(u,z)$ converges absolutely for $|u| < q^{-\frac{1}{h+1}}$ and $|z| \leq A$. By Cauchy's integral formula over $|u| = q^{-\frac{1+\varepsilon}{h+1}}$, we obtain

$$b_z(P^j) = \frac{1}{2\pi i}\oint_{|u|=q^{-\frac{1+\varepsilon}{h+1}}}(1 + zu + u^2 + \cdots + u^{h-1})(1-u)^z\frac{du}{u^{j+1}}.$$

Thus,

$$|b_z(P^j)| \leq q^{\frac{j}{h+1}(1+\varepsilon)}M_A,$$

for $j \geq 2$, where

$$M_A := \sup_{|z|\leq A, |u|\leq\left(\frac{2}{3}\right)^{\frac{1}{h+1}}}\left|(1 + zu + u^2 + \cdots + u^{h-1})(1-u)^z\right|$$

is a constant depending on A. The rest of the proof proceeds exactly as in [11, Lemma 3.1] to obtain

$$\sum_{0\leq a\leq n}\frac{|\mathcal{B}_z(a)|}{q^{\sigma a}} \ll_{\sigma.q} \exp\left(\frac{M_A}{q^{2\sigma-1}-1}\right),$$

where the implied constant is independent of n. □

Since $a^{2A+2} < q^{a/3}$ as a approaches infinity, it follows from Lemma 2 that

$$\sum_{a\geq 0} \frac{|\mathcal{B}_z(a)|}{q^a} a^{2A+2} < \sum_{a\geq 0} \frac{|\mathcal{B}_z(a)|}{q^{\frac{2a}{3}}} \ll_A 1$$

uniformly for $|z| \leq A$. Thus we can apply Theorem 11 to $\mathcal{B}_{1,\mathcal{S}_h}(u,z)$.

Proof of Theorem 5. Recall that we have

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}} z^{\omega_h(f)} u^{\deg(f)} = \mathcal{C}_{h,\mathcal{N}_h}(u,z) = \mathcal{B}_{h,\mathcal{N}_h}(u,z)\mathcal{Z}_q(u^h)^z.$$

Applying Theorem 11, this gives

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} z^{\omega_h(f)} = \frac{q^{\frac{n}{h}} n^{z-1}}{h^z\Gamma(z)} \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}, z\right) + O_A(q^{\frac{n}{h}} n^{\mathrm{Re}(z)-2}). \tag{35}$$

Differentiating both sides of Equation (35) with respect to z for z close to 1 and applying Equation (11) we get:

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} \omega_h(f) z^{\omega_h(f)-1} =& \frac{q^{\frac{n}{h}} n^{z-1}}{h^z} \log\left(\frac{n}{h}\right) \sum_{j=0}^{h-1} \xi_h^{jn} \frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}, z\right)}{\Gamma(z)} \\ &+ \frac{q^{\frac{n}{h}} n^{z-1}}{h^z} \sum_{j=0}^{h-1} \xi_h^{jn} \left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}, z\right)}{\Gamma(z)}\right)' \\ &+ O_z(1) O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned} \tag{36}$$

Evaluating Equation (36) at $z = 1$ we have

$$\begin{aligned}&\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} \omega_h(f) \\ &= \frac{q^{\frac{n}{h}}}{h} \log\left(\frac{n}{h}\right) \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1\right) \\ &+ \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \frac{\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1)\Gamma(1) - \mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1)\Gamma'(1)}{\Gamma(1)^2} \\ &+ O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned} \tag{37}$$

Note that we have

$$\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1) = \prod_P \left(1 - \frac{1}{|P|}\right)\left(1 + \frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right). \tag{38}$$

The logarithmic derivative of $\mathcal{B}_{h,\mathcal{N}_h}(u,z)$ gives

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}(u,z)}{\mathcal{B}_{h,\mathcal{N}_h}(u,z)} = \sum_P \Bigg(\log(1-u^{h\deg(P)}) + \frac{u^{h\deg(P)}(1-u^{\deg(P)})}{1-u^{\deg(P)}+u^{h\deg(P)}+(z-1)u^{h\deg(P)}(1-u^{\deg(P)})}\Bigg), \quad (39)$$

and thus

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)}{\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)} = \sum_P \left(\log\left(1-\frac{1}{|P|}\right) + \frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right). \quad (40)$$

Substituting the above result and Equation (38) in Equation (37) gives

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} \omega_h(f) &= \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\\ &\times\left[\log\left(\frac{n}{h}\right)+\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)+\gamma\right]\\ &+O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned}$$

Combining the above results with the definition of B_1 proves Equation (8).

We now proceed to prove Equation (9). Multiplying Equation (36) by z, differentiating both sides with respect to z for z close to 1, and applying Equation (12), we obtain

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} \omega_h(f)^2 z^{\omega_h(f)-1} &= \left(1+z\log\left(\frac{n}{h}\right)\right)\frac{q^{\frac{n}{h}}n^{z-1}}{h^z}\log\left(\frac{n}{h}\right)\sum_{j=0}^{h-1}\xi_h^{jn}\frac{\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)}{\Gamma(z)}\\ &+\left(1+2z\log\left(\frac{n}{h}\right)\right)\frac{q^{\frac{n}{h}}n^{z-1}}{h^z}\sum_{j=0}^{h-1}\xi_h^{jn}\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)}{\Gamma(z)}\right)' \quad (41)\\ &+\frac{q^{\frac{n}{h}}n^{z-1}}{h^z}z\sum_{j=0}^{h-1}\xi_h^{jn}\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)}{\Gamma(z)}\right)''+O_z(1)O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).\end{aligned}$$

Now notice that

$$\begin{aligned}
&\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)}{\Gamma(z)}\right)'' \\
&\quad= \left(\frac{\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)\Gamma(z) - \mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)\Gamma'(z)}{\Gamma(z)^2}\right)' \\
&\quad= \frac{1}{\Gamma(z)^3}\Bigg[\frac{\partial^2}{\partial z^2}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)\Gamma(z)^2 - 2\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)\Gamma(z)\Gamma'(z) \\
&\qquad + \mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},z)(2\Gamma'(z)^2 - \Gamma(z)\Gamma''(z))\Bigg]. \qquad (42)
\end{aligned}$$

Differentiating from Equation (39), we have that

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1) \\
&\quad= \frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\sum_P\left(\log\left(1-\frac{1}{|P|}\right) + \frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right) \\
&\qquad - \mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\sum_P\left(\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)^2 \\
&\quad= \prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right) \\
&\qquad \times\Bigg[\left(\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)\right)^2 \\
&\qquad - \sum_P\left(\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)^2\Bigg]. \qquad (43)
\end{aligned}$$

Evaluating Equation (41) at $z=1$, recalling that $\Gamma(1)=1$, $\Gamma'(1)=-\gamma$, $\Gamma''(1)=\gamma^2+\zeta(2)$, and inserting Equation (42) in Equation (41) gives

$$\begin{aligned}
&\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega_h(f)^2 \\
&\quad= \left(1+\log\left(\frac{n}{h}\right)\right)\frac{q^{\frac{n}{h}}}{h}\log\left(\frac{n}{h}\right)\sum_{j=0}^{h-1}\xi_h^{jn}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1) + \left(1+2\log\left(\frac{n}{h}\right)\right) \\
&\qquad \times\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\left[\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)+\gamma\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\right] \\
&\qquad + \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\Bigg[\frac{\partial^2}{\partial z^2}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)+2\gamma\frac{\partial}{\partial z}\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1) \\
&\qquad + (\gamma^2-\zeta(2))\mathcal{B}_{h,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\Bigg] + O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).
\end{aligned}$$

Inserting Equations (38), (40), and (43) in the above expression gives

$$
\begin{aligned}
&\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega_h(f)^2\\
&=\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\\
&\times\Bigg[\left(\log\left(\frac{n}{h}\right)\right)^2+\log\left(\frac{n}{h}\right)\\
&\times\left[1+2\gamma+2\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)\right]\\
&+\left(\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)+\gamma\right)^2\\
&+\sum_P\left(\log\left(1-\frac{1}{|P|}\right)+\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)+\gamma-\zeta(2)\\
&-\sum_P\left(\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)}}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)^2\Bigg]\\
&+O_\varepsilon\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right).
\end{aligned}
$$

Combining the above with the definition of B_1 proves Equation (9).

An explicit expression for $|\mathcal{N}_h\cap\mathcal{M}_n|$ can be found in [11, Lemma 5.3]. Indeed, for $n\geq h$, we have

$$
\begin{aligned}
&|\mathcal{N}_h\cap\mathcal{M}_n|\\
&=\frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)+O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right).
\end{aligned}
$$

The variance can be then directly computed by combining the above with Equations (8) and (9). □

4.2. First and Second Moments of ω_k for h-Full Polynomials

Here we consider the case $k>h$. We have that $\mathcal{C}_{k,\mathcal{S}_h}(u,z)$ has poles of order 1 when $u=\frac{1}{q^h}$. We extract these poles as

$$
\begin{aligned}
&\mathcal{B}_{k,\mathcal{N}_h}(u,z)=\mathcal{Z}_q(u^h)^{-1}\mathcal{C}_{k,\mathcal{S}_h}(u,z)\\
&=\prod_P\left(1+\frac{u^{(h+1)\deg(P)}(1-u^{(h-1)\deg(P)})}{1-u^{\deg(P)}}+(z-1)u^{k\deg(P)}(1-u^{h\deg(P)})\right),
\end{aligned}
$$

where $\mathcal{B}_{k,\mathcal{N}_h}(u,z)$ is absolutely convergent for $|u| < q^{-\frac{1}{h+1}}$ and $|z| \le A$.

We use Perron's formula (Theorem 9) to compute the first moment and obtain

$$\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} z^{\omega_k(f)} = \frac{1}{2\pi i} \oint \frac{\mathcal{B}_{k,\mathcal{N}_h}(u,z)}{(1-qu^h)u^n} \frac{du}{u},$$

where the integral takes place in a small circle around the origin. We move the circle to $|u| = q^{-\varepsilon - \frac{1}{h+1}}$ and obtain the residues at $u = (q^{\frac{1}{h}} \xi_h^j)^{-1}$. This gives

$$\begin{aligned}\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} z^{\omega_k(f)} &= -\sum_{j=0}^{h-1} \operatorname{Res}_{u=(q^{\frac{1}{h}}\xi_h^j)^{-1}} \frac{\mathcal{B}_{k,\mathcal{N}_h}(u,z)}{(1-qu^h)u^{n+1}} \\ &\quad + \frac{1}{2\pi i} \oint_{|u|=q^{-\varepsilon+\frac{1}{h+1}}} \frac{\mathcal{B}_{k,\mathcal{N}_h}(u,z)}{(1-qu^h)u^n} \frac{du}{u} \\ &= \sum_{j=0}^{h-1} (q^{\frac{1}{h}} \xi_h^j)^{n+1} \mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, z) \frac{1}{q^{\frac{1}{h}} \xi_h^j \prod_{m \neq j} \left(1 - \xi_h^{m-j}\right)} \\ &\quad + O_z(1) O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right) \\ &= \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, z) + O_z(1) O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right). \end{aligned} \quad (44)$$

Proof of Theorem 7. To recover the first moment, we differentiate and evaluate the above equation at $z = 1$. This gives

$$\sum_{f \in \mathcal{N}_h \cap \mathcal{M}_n} \omega_k(f) = \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \frac{\partial}{\partial z} \mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1) + O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right). \quad (45)$$

We have

$$\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1) = \prod_P \left(1 - \frac{1}{|P|}\right)\left(1 + \frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right). \quad (46)$$

The logarithmic derivative gives

$$\begin{aligned}&\frac{\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{N}_h}(u,z)}{\mathcal{B}_{k,\mathcal{N}_h}(u,z)} \\ &\quad = \sum_P \frac{u^{k\deg(P)}(1-u^{\deg(P)})}{(1-u^{\deg(P)}+u^{h\deg(P)}) + (z-1)u^{k\deg(P)}(1-u^{\deg(P)})},\end{aligned}$$

and thus

$$\frac{\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1)}{\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1}, 1)} = \sum_P \frac{|P|(q^{\frac{1}{h}}\xi_h^j)^{-k\deg(P)}(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}. \quad (47)$$

Applying the above results to Equation (45) gives the first moment Equation (10).

We proceed to compute the second moment. Differentiating Equation (44), multiplying by z, differentiating again, and setting $z=1$, we have

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\omega_k(f)^2 = \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\left(\frac{\partial^2}{\partial z^2}\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)+\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\right) + O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right). \tag{48}$$

For the second derivative we have

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2}\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\\
&\quad=\frac{\partial}{\partial z}\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\sum_P\frac{|P|(q^{\frac{1}{h}}\xi_h^j)^{-k\deg(P)}(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\\
&\quad-\mathcal{B}_{k,\mathcal{N}_h}((q^{\frac{1}{h}}\xi_h^j)^{-1},1)\sum_P\left(\frac{|P|(q^{\frac{1}{h}}\xi_h^j)^{-k\deg(P)}(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)^2\\
&\quad=\prod_P\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}\right)\\
&\quad\times\left[\left(\sum_P\frac{|P|(q^{\frac{1}{h}}\xi_h^j)^{-k\deg(P)}(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}{|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1}\right)^2\right.\\
&\quad\left.-\sum_P\left(\frac{|P|(q^{\frac{1}{h}}\xi_h^j)^{-k\deg(P)}(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})}{(|P|(1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P)})+1)}\right)^2\right].
\end{aligned}$$

Combining this with Equations (46) and (47) in Equation (48), we get the desired result. □

4.3. Normal Order and an Erdős–Kac Result for h-Full Polynomials

The goal of this section is to prove Theorems 6 and 8. We start by proving Theorems 6, which in particular implies that ω_h has normal order over the h-full polynomials.

Proof of Theorem 6. Our argument follows very closely the proof of Theorem 2 and [11, Theorem 1.6]. We will prove that, as $n\to\infty$,

$$\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\left(\frac{\omega_h(f)-\log\left(\frac{n}{h}\right)}{\sqrt{\log\left(\frac{n}{h}\right)}}\right)^v\to C_v,$$

where C_v is given by Equation (27).

As before we will consider the moment generating function evaluated at $z = e^t$,

$$\mathcal{C}_{h,\mathcal{N}_h}(u,e^t) = \sum_{f\in\mathcal{N}_h\cap\mathcal{M}} e^{t\omega_k(f)}u^{\deg(f)} = \prod_P (1 + e^t u^{h\deg(P)} + u^{(h+1)\deg(P)} + \cdots),$$

and we extract the singularities at $u^h = \frac{1}{q}$ as

$$\mathcal{B}_{h,\mathcal{N}_h}(u,e^t) = \mathcal{Z}_q(u^h)^{-e^t}\mathcal{C}_{h,\mathcal{N}_h}(u,e^t).$$

By applying Theorem 11 we get

$$\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n} e^{\omega_1(f)t} = \frac{q^{\frac{n}{h}}n^{e^t-1}}{h^{e^t}}\sum_{s=0}^{h-1}\frac{\xi_h^{sn}}{\Gamma(e^t)}\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right) + O_A(q^{\frac{n}{h}}n^{\mathrm{Re}(e^t)-2}).$$

By considering the moment generating function, and combining with Equation (30), we have, as in the h-free case,

$$\mathbb{E}\left(\omega_h^\ell\right) = \left(\sum_{s=0}^{h-1}\xi_h^{sn}\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}\sum_{j=0}^{\ell}\binom{\ell}{j}h^jT_j\left(\log\left(\frac{n}{h}\right)\right) \tag{49}$$
$$\times\sum_{s=0}^{h-1}\xi_h^{sn}\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^t)}\right)^{(\ell-j)}\Bigg|_{t=0} + O_\varepsilon\left(\frac{1}{n^{1-\varepsilon}}\right).$$

Notice that

$$\frac{1}{|\mathcal{N}_h\cap\mathcal{M}_n|}\sum_{f\in\mathcal{N}_h\cap\mathcal{M}_n}\left(\frac{\omega_h(f)-\log\left(\frac{n}{h}\right)}{\sqrt{\log\left(\frac{n}{h}\right)}}\right)^v$$
$$= \frac{1}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\omega_h^\ell)(-1)^{v-\ell}\left(\log\left(\frac{n}{h}\right)\right)^{v-\ell}.$$

Combining with Equation (49), we then have

$$\frac{1}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\omega_h^\ell)(-1)^{v-\ell}\left(\log\left(\frac{n}{h}\right)\right)^{v-\ell}$$
$$= \frac{\left(\sum_{s=0}^{h-1}\xi_h^{sn}\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\sum_{j=0}^{\ell}\binom{\ell}{j}T_j\left(\log\left(\frac{n}{h}\right)\right) \tag{50}$$
$$\times\sum_{s=0}^{h-1}\xi_h^{sn}\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^t)}\right)^{(\ell-j)}\Bigg|_{t=0}(-1)^{v-\ell}\left(\log\left(\frac{n}{h}\right)\right)^{v-\ell}$$
$$+ O_\varepsilon\left(\frac{1}{n^{1-\varepsilon}}\right).$$

Consider the change of variables $u = v - \ell$, $m = v - \ell + j$. Then the main term in Equation (50) becomes

$$\begin{aligned}
&\frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}} \\
&\qquad \times \sum_{m=0}^{v}\sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^{ht})}\right)^{(v-m)}\Bigg|_{t=0} \\
&\qquad \times \sum_{u=0}^{m}\binom{v}{u}\binom{v-u}{m-u} T_{m-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(\log\left(\frac{n}{h}\right)\right)^u \\
&\quad = \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}} \\
&\qquad \times \sum_{m=0}^{v}\binom{v}{m}\sum_{s=0}^{h-1} \xi_h^{sn} \left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},e^t\right)}{\Gamma(e^{ht})}\right)^{(v-m)}\Bigg|_{t=0} \\
&\qquad \times \sum_{u=0}^{m}\binom{m}{u} T_{m-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(\log\left(\frac{n}{h}\right)\right)^u.
\end{aligned}$$

Similarly to the h-free case we get, for v even, that the main term should come from setting $m = v$, which leads to

$$\begin{aligned}
&\frac{1}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}}\sum_{\ell=0}^{v}\binom{v}{\ell}\mathbb{E}(\omega_h(f)^\ell)(-1)^{v-\ell}\left(\log\left(\frac{n}{h}\right)\right)^{v-\ell} \\
&\quad = \frac{\left(\sum_{s=0}^{h-1} \xi_h^{sn} \mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)\right)^{-1}}{\left(\log\left(\frac{n}{h}\right)\right)^{\frac{v}{2}}}\sum_{s=0}^{h-1} \xi_h^{sn}\left(\frac{\mathcal{B}_{h,\mathcal{N}_h}\left((q^{\frac{1}{h}}\xi_h^s)^{-1},1\right)}{\Gamma(1)}\right) \\
&\qquad \times \sum_{u=0}^{v}\binom{v}{u} T_{v-u}\left(\log\left(\frac{n}{h}\right)\right)(-1)^u\left(\log\left(\frac{n}{h}\right)\right)^u + O\left(\frac{1}{\log n}\right) \\
&\quad = \frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!} + O\left(\frac{1}{\log n}\right),
\end{aligned}$$

while for v odd we get

$$O\left(\frac{1}{\sqrt{\log n}}\right).$$

□

Before proceeding to the proof of Theorem 8 we need the following auxiliary result.

Lemma 5. *Let $n \geq \ell > h > 0$ be integers. Then*

$$|\mathcal{N}_h \cap \mathcal{M}_n \cap \mathcal{S}_\ell| = \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}\left((q^{\frac{1}{h}} \xi_h^j)^{-1}\right) + O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right), \tag{51}$$

where $\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u)$ is defined below by Equation (53). In addition, let $P_0 \in \mathcal{P}$ be fixed. Then

$$\begin{aligned}\sum_{\substack{f\in\mathcal{N}_h\cap\mathcal{M}_n\cap\mathcal{S}_\ell \\ (f,P_0)=1}} 1 =& \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}\left((q^{\frac{1}{h}} \xi_h^j)^{-1}\right) \\ &\times \left(\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P_0)}}{1-(q^{\frac{1}{h}}\xi_h^j)^{-\deg(P_0)} + (q^{\frac{1}{h}}\xi_h^j)^{-h\deg(P_0)} - (q^{\frac{1}{h}}\xi_h^j)^{-\ell\deg(P_0)}}\right) \\ &+ O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right).\end{aligned} \tag{52}$$

Proof. We consider the generating series for the polynomials that are simultaneously h-full and k-free,

$$\begin{aligned}\mathcal{G}_{\mathcal{N}_h,\mathcal{S}_\ell}(u) := \sum_{f\in\mathcal{N}_h\cap\mathcal{S}_\ell\cap\mathcal{M}} u^{\deg(f)} =& \prod_P \left(1 + u^{h\deg(P)} + \cdots + u^{(\ell-1)\deg(P)}\right) \\ =& \prod_P \left(1 + \frac{u^{h\deg(P)} - u^{\ell\deg(P)}}{1-u^{\deg(P)}}\right).\end{aligned}$$

We extract the poles at $u^h = \frac{1}{q}$ as follows,

$$\begin{aligned}\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u) =& \mathcal{Z}_q(u^h)^{-1}\mathcal{G}_{\mathcal{N}_h,\mathcal{S}_\ell}(u) \\ =& \prod_P \left(1 + \frac{u^{(h+1)\deg(P)} - u^{\ell\deg(P)} - u^{2h\deg(P)} + u^{(h+\ell)\deg(P)}}{1-u^{\deg(P)}}\right),\end{aligned} \tag{53}$$

where $\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u)$ is absolutely convergent for $|u| < q^{-\frac{1}{h+1}}$.

By Perron's formula (Theorem 9), and by moving the integral to $|u| = q^{-\varepsilon-\frac{1}{h+1}}$,

$$\begin{aligned}\sum_{f\in\mathcal{N}_h\cap\mathcal{S}_\ell\cap\mathcal{M}_n} 1 =& \frac{1}{2\pi i} \oint \frac{\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u)}{1-qu^h} \frac{du}{u^{n+1}} \\ =& -\sum_{j=0}^{h-1} \operatorname{Res}_{u=(q^{\frac{1}{h}}\xi_h^j)^{-1}} \frac{\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u)}{(1-qu^h)u^{n+1}} \\ &+ \frac{1}{2\pi i} \oint_{|u|=q^{-\varepsilon-\frac{1}{h+1}}} \frac{\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}(u)}{1-qu^h} \frac{du}{u^{n+1}} \\ =& \frac{q^{\frac{n}{h}}}{h} \sum_{j=0}^{h-1} \xi_h^{jn} \mathcal{H}_{\mathcal{N}_h,\mathcal{S}_\ell}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}\right) + O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right).\end{aligned}$$

This gives a proof of Equation (51). To prove Equation (52), we can proceed similarly, but using the generating series

$$\mathcal{G}^{P_0}_{\mathcal{N}_h,\mathcal{S}_\ell}(u) := \sum_{\substack{f\in\mathcal{N}_h\cap\mathcal{S}_\ell\cap\mathcal{M}\\(f,P_0)=1}} u^{\deg(f)} = \prod_{P\neq P_0}\left(1+\frac{u^{h\deg(P)}-u^{\ell\deg(P)}}{1-u^{\deg(P)}}\right)$$

instead. □

Proof of Theorem 8. As in the proof of Theorem 4, let $G(f)$ be a nondecreasing function $G:\mathcal{N}_h\cap\mathcal{M}\to\mathbb{R}_{\geq 0}$. First assume that there is an $f_0\in\mathcal{N}_h\cap\mathcal{M}$ such that $G(f_0)>0$. Therefore $G(f)>0$ for all $f\in\mathcal{N}_h\cap\mathcal{M}$ such that $\deg(f)>\deg(f_0)$. Let $n>\deg(f_0)$ and consider the following set:

$$\mathcal{O}_{\mathcal{N},0}(n,h) := \{f\in\mathcal{N}_h\cap\mathcal{M}_n \ : \ \omega_k(f)=0\}.$$

It can be seen that

$$\mathcal{N}_h\cap\mathcal{M}_n\cap\mathcal{S}_k\subseteq\mathcal{O}_{\mathcal{N},0}(n,h),$$

and therefore

$$|\mathcal{O}_{\mathcal{N},0}(n,h)|\geq|\mathcal{N}_h\cap\mathcal{M}_n\cap\mathcal{S}_k| = \frac{q^{\frac{n}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{jn}\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_k}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}\right)+O_\varepsilon\left(q^{\frac{n}{h+1}+\varepsilon n}\right)$$

by Lemma 5.

Since h and k are fixed, this means that $|\mathcal{O}_{\mathcal{N},0}(n,h)|$ is $\gg q^{\frac{n}{h}}$. As in the case of the proof of Theorem 4, we conclude that $\omega_k(f)$ does not have normal order G when G is not the constant function 0.

Now, if $G(f)=0$ for all $f\in\mathcal{N}_h\cap\mathcal{M}$ we define

$$\mathcal{O}_{\mathcal{N},1}(n,h) := \{f\in\mathcal{N}_h\cap\mathcal{M}_n \ : \ \omega_k(f)=1\}.$$

Let $P\in\mathcal{P}_1$ be a fixed monic irreducible polynomial of degree 1. It can be seen that:

$$\begin{aligned}|\mathcal{O}_{\mathcal{N},1}(n,h)| &\geq \sum_{\substack{f\in\mathcal{N}_h\cap\mathcal{M}_n,\,\nu_P(f)=k\\ \nu_Q(f)<k,\forall Q\in\mathcal{P},Q\neq P}} 1\\ &\geq \sum_{\substack{f\in\mathcal{N}_h\cap\mathcal{M}_{n-k}\cap\mathcal{S}_k\\(f,P)=1}} 1\\ &=\frac{q^{\frac{n-k}{h}}}{h}\sum_{j=0}^{h-1}\xi_h^{j(n-k)}\mathcal{H}_{\mathcal{N}_h,\mathcal{S}_k}\left((q^{\frac{1}{h}}\xi_h^j)^{-1}\right)\\ &\quad\times\left(\frac{1-(q^{\frac{1}{h}}\xi_h^j)^{-1}}{1-(q^{\frac{1}{h}}\xi_h^j)^{-1}+(q^{\frac{1}{h}}\xi_h^j)^{-h}-(q^{\frac{1}{h}}\xi_h^j)^{-k}}\right)+O_\varepsilon\left(q^{\frac{n-k}{h+1}+\varepsilon n}\right),\end{aligned}$$

where we have applied Lemma 5.

Since h and k are fixed, this means that $|\mathcal{O}_{\mathcal{N},1}(n,h)|$ is $\gg q^{\frac{n}{h}}$. As in the case of Theorem 4, we conclude that $\omega_k(f)$ does not have normal order G when G is the constant function 0. □

5. Conclusion

This work represents a natural merging of questions from both [3] and [11]. Namely, we have taken the functions ω_k from [3], which are refinements of the number of distinct prime factors function ω, and have considered them over the subfamilies of h-free and h-full polynomials as in [11]. The current results support the findings of the previous works. We recall from [11] that a motivation for studying and comparing results in these two families of polynomials lies in the fact that the h-free polynomials represent a positive proportion of the whole polynomial family, while the h-full polynomials do not, as their size is of order $q^{\frac{n}{h}}$, while the size of the full family is of order q^n. It is therefore more surprising that the h-full polynomials satisfy an Erdős-Kac type of result than the h-free polynomials satisfy such result. From this work, we now conclude that the weight of this behaviour is carried by ω_1 for the h-free polynomials, and by ω_h for the h-full polynomials. This phenomenon is not so surprising for ω_1 and the h-free polynomials, since it is exactly as observed in [3]. However, the case of ω_h and the h-full polynomials is less immediate to predict.

The most evident direction for extending this work is to consider the number field case, naturally restricting the results of [7] to h-free and h-full numbers. Das, Kuo, and Liu have informed us that they are pursuing this direction.

Other directions of future research could include the study of intersection sets of the form $\mathcal{N}_h \cap \mathcal{S}_k$, as well as more general settings, such as considering polynomials f satisfying that $\nu_P(f)$ belongs to a union of some fixed intervals.

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