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ON THE SIDON IDEAL

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Abstract

We investigate the ideals of the natural numbers generated by Sidon sets and weak Sidon sets, proving that they are, in fact, the same. We examine several properties of the resulting ideal and place this ideal in the Kowitz Diagram of ideals.

1. Introduction

In recent years, we have witnessed the rapid development of the theory of ideals. Many examples of ideals have already been considered; some of them originate from number theory. Based on the well-known concept of a Sidon set, we define a new ideal — namely the Sidon ideal — and determine its place within the family of other known ideals. We denote by $\mathbb{N}$ the set of all natural numbers with the number 0 adjoined. For infinite sets $A, B \subseteq \mathbb{N}$ we write $A \subseteq^* B$ if and only if $A \setminus B$ is finite. If $A \subseteq \mathbb{N}$ is an infinite set, then $A(n)$ denotes the n-th element of A. If $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, then $A + n = \{a + n \colon a \in A\}$. If A, B are nonempty subsets of the natural numbers, then we define $dist(A, B) = \min\{|a - b| \colon a \in A, b \in B\}$. Also, recall the following standard notation (see for example [3]): If $A, B \subseteq \mathbb{N}$ then define $A - B = \{a - b \colon a > b, a \in A, b \in B\}$ and $D(A) = A - A$.

Definition 1. Suppose that $A \subseteq \mathbb{N}$. We define $\mathrm{CenSym}(A)$ by

$$\{n \in A \colon \text{ there exists } h \in \mathbb{N}, h > 0 \text{ such that } n - h, n + h \in A\}.$$

An *ideal* is a nonempty family $\mathcal{I} \subset P(\mathbb{N})$ closed under taking subsets and finite unions, i.e.,

1. $\emptyset \in \mathcal{I}$;
2. for all $A \in \mathcal{I}$ and all $B \subset A$, we have $B \in \mathcal{I}$;
3. for all $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$.

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An ideal is *proper* if $\mathbb{N} \notin \mathcal{I}$, or, equivalently, $\mathcal{I} \neq P(\mathbb{N})$. Unless stated otherwise, we assume all ideals are proper and we assume also that all singletons belong to the ideal. An ideal is *dense* if every infinite subset of $\mathbb{N}$ contains an infinite subset belonging to the ideal. An ideal $\mathcal{I}$ is called a *P-ideal* if and only if, for each sequence $A_0 \subseteq^* A_1 \subseteq^* \cdots$ of sets from $\mathcal{I}$, there exists $A^* \in \mathcal{I}$ such that $A_n \subseteq A^*$ for all n. The following four definitions are taken from [6].

Definition 2. The *difference ideal* $\mathcal{D}$ we define by

$$\mathcal{D} = \{A \subseteq \mathbb{N} \colon \text{ for all infinite } Z \subseteq \mathbb{N}, D(Z) \not\subseteq A\}.$$

Definition 3. The $\mathcal{D}_{\mathit{fin}}$ *ideal* we define by

$$\mathcal{D}_{\mathit{fin}} = \{A \subseteq \mathbb{N} \colon \text{ there exists } n \in \mathbb{N} \text{ such that for all } Z \in [\mathbb{N}]^n, D(Z) \not\subseteq A\}.$$

Definition 4. An infinite set $A \subseteq \mathbb{N}$ is said to be a *lacunary set* if and only if $\lim_{n\to\infty} A(n+1) - A(n) = \infty$.

Definition 5. The *Lacunary ideal* is the ideal generated by lacunary subsets and we denote it by $\mathcal{L}ac$.

Definition 6. An infinite set $A \subseteq \mathbb{N}$ is a *thin* set if and only if $\lim_{n\to\infty} \frac{A(n)}{A(n+1)} = 0$.

Definition 7. An infinite set $A \subseteq \mathbb{N}$ is an *almost thin* set if and only if

$$\limsup_{n\to\infty} \frac{A(n)}{A(n+1)} < 1.$$

Definition 8. The ideal generated by thin sets is called *the thin sets ideal* and we denote it by $\mathcal{T}$. The ideal generated by almost thin sets is called *the almost thin sets ideal* and we denote it by $\mathcal{A}$.

Definition 9. Let E_n denote the family of subsets of natural numbers which do not contain any arithmetic sequence of length n. Let us define W as the family of subsets of natural numbers which do not contain arithmetic sequences of arbitrary length n:

$$\begin{aligned} \mathrm{W} = \{A \subseteq \mathbb{N} \colon &\text{ there exists } n > 0 \text{ such that for all } a \in \mathbb{N}, r > 0 \\ &\text{there exists } j < n \text{ such that } a + r \cdot j \notin A\}. \end{aligned}$$

This is an ideal and we call it the *van der Waerden ideal*.

2. The Sidon Ideal

Let us recall the main tool of our paper.

Definition 10. A subset (sequence) $A \subseteq \mathbb{N}$ is called a *Sidon set* if and only if for $n, m \in A$, $n \leq m$, all sums $n + m$ are distinct, i.e., for all $n, m, k, l \in A$, if $n \leq m, k \leq l$, and $n + m = k + l$, then $n = k$.

The following definition is taken from [5].

Definition 11. A subset (sequence) $A \subseteq \mathbb{N}$ is called a *weak Sidon set* (or *well-spread*) if for $n, m \in A$, $n < m$, all sums $n + m$ are distinct, i.e., for all $n, m, k, l \in A$, if $n < m, k < l$, and $n + m = k + l$, then $n = k$.

Recall here that the notion of Sidon sets appeared also in [3], page 6, and in [6] under the name $\mathcal{D}$-sparse sets.

Definition 12. A set $A \subseteq \mathbb{N}$ is *$\mathcal{D}$-sparse* if for every $k \in D(A)$ there is only one pair $n, m \in A$ with $k = m - n$.

Notice that this notion is equivalent to the so-called "Golomb ruler" (in fact, Golomb ruler is a finite version of $\mathcal{D}$-sparse sets), see for example [2]. A simple computation shows that the notions of $\mathcal{D}$-sparse sets and Sidon sets are in fact identical.

Let us define the main notion of this paper.

Definition 13. The *Sidon ideal*, denoted by $\mathcal{S}i$, is the ideal generated by the family of all Sidon sets, i.e.,

$$\mathcal{S}i = \{A \subseteq \mathbb{N}\colon \text{ there exists Sidon sets } A_1, \ldots, A_n \text{ such that } A \subseteq \bigcup_{j=1}^{n} A_j\}.$$

It is also tempting to define "a weak Sidon ideal" as an ideal generated by all weak Sidon sets, but this ideal coincides with $\mathcal{S}i$.

Theorem 1. *Every weak Sidon set can be partitioned into two Sidon sets.*

Proof. It is well known that, if A is a weak Sidon set, then $A \setminus \mathrm{CenSym}(A)$ is a Sidon set. This is because the only equality in the definition of a Sidon set that shows that a weak Sidon set A is not a Sidon set is the equation $n + m = 2 \cdot k$, $n < m$, where $n \neq k$, and this is the case when $k \in \mathrm{CenSym}(A)$. Now, assume that A is a finite weak Sidon set. Define a sequence by setting $A_0 := A$; $A_{n+1} := \mathrm{CenSym}(A_n)$. This sequence is decreasing and since A is a finite set there exists n_0 such that, for all $n > n_0$, we have $A_n = \emptyset$ (for a finite B we cannot have $\mathrm{CenSym}(B) = B$ unless $B = \emptyset$). Let us define $E(A) = \bigcup_{k=0}^{\infty} A_{2k+1} \setminus A_{2k+2}$ and $O(A) = \bigcup_{k=0}^{\infty} A_{2k} \setminus A_{2k+1}$. Of course, $\{E(A), O(A)\}$ is a partition of A.

We claim that the sets $E(A)$ and $O(A)$ are Sidon sets. To prove the claim, suppose by way of contradiction that there exist n and $h > 0$ such that $n - h, n, n + h \in E(A)$. Then $n \in A_{2k_0+1} \setminus A_{2k_0+2}$ for some k_0. Since $n \in \mathrm{CenSym}(A_{2k_0})$, there exists $h_1 > 0$ such that $n - h_1, n, n + h_1 \in A_{2k_0}$. It is clear that, since A is a weak Sidon set, we have $h = h_1$. Hence $n - h, n + h \in A_{2k_0}$. Now, suppose that we have $n - h, n + h \in A_{2k_0+1}$. Then we would have $n, n - h, n + h \in A_{2k_0+1}$ and this would imply that $n \in \mathrm{CenSym}(A_{2k_0+1})$, so $n \in A_{2k_0+2}$, which would be impossible. Thus, either $n - h \notin A_{2k_0+1}$ or $n + h \notin A_{2k_0+1}$. Suppose for example that $n - h \notin A_{2k_0+1}$. Then $n - h \in A_{2k_0} \setminus A_{2k_0+1}$, so $n - h \in O(A)$ which is a contradiction. This proves the claim.

Thus, we have proved that each finite weak Sidon set can be partitioned into two Sidon sets. Using a standard compactness argument, we conclude that any infinite weak Sidon set can be partitioned into two Sidon sets. □

Corollary 1. *The ideal generated by weak Sidon sets is the same as the ideal generated by Sidon sets.*

Note that the collection $\mathcal{S}$ of all Sidon sets $A \subseteq \mathbb{N}$ is a perfect subset of the Cantor space $2^{\mathbb{N}}$.

Corollary 2. *The Sidon ideal is an F_σ ideal.*

Proof. Observe that the function $\cup_n \colon (2^\omega)^n \to 2^\omega$, defined by $\cup_n(A_1, \ldots, A_n) = \bigcup_{j=1}^n A_j$, is continuous, and $\mathcal{S}i = \bigcup_{n=1}^\infty \cup_n[\mathcal{S}^n]$. □

3. The Place of the Sidon Ideal in the Kowitz Diagram

It is proved in [3] that (see Proposition 4.3) for each infinite $A \subseteq \mathbb{N}$ there exists an infinite $\mathcal{D}$-sparse $B \subseteq A$. This shows that the Sidon ideal is dense. Moreover, the author proves (see Proposition 4.3 (1)) that if A is $\mathcal{D}$-sparse and $n \in \mathbb{N}$ then $A + n$ is in the difference ideal. In particular, this shows that the Sidon ideal is contained in the difference ideal.

Let us prove a little more.

Theorem 2. *The Sidon ideal is contained in the lacunary ideal.*

Proof. Suppose that $S \subseteq \mathbb{N}$ is a Sidon set, and let $M > 0$. For any $m \leq M$ there exists at most one pair $\{S(n), S(n+1)\}$ such that $S(n+1) - S(n) = m$. If there exists such a pair, define $n_m = n$. If there is no such pair, define $n_m = 0$. Define $N = \max\{n_m \colon m \leq M\} + 1$. Then, for $n > N$ we have $S(n+1) - S(n) > M$, so S is a lacunary set. □

It is straightforward that the Sidon ideal is included in the Van der Waerden ideal. This is because any Sidon set cannot contain a 3-element increasing arithmetic sequence. Recall the following results concerning the existence of so-called "fat" Sidon set.

Theorem 3. *There exists an infinite Sidon set A such that*

1. $|[1,n] \cap A| > c\sqrt[3]{n}$ *for some* $c > 0$ *(see [7]);*
2. $|[1,n] \cap A| > c\sqrt[3]{n\log(n)}$ *(see [1]);*
3. $|[1,n] \cap A| > n^{\sqrt{2}-1-o(1)}$ *(see [8]).*

Even the first example of Theorem 3 suffices to prove the following proposition, which is a construction of a special Sidon set.

Proposition 1. *There exists a Sidon set A which is not in the ideal $\mathcal{A}$.*

Proof. Let us begin with the following (probably well known) asymptotic behavior of any almost thin set. Suppose that $A \subseteq \mathbb{N}$ is an almost thin set. Then there are $C, D > 0$ such that $|[0,M) \cap A| \leq C\log M + D$. Let us sketch the proof. There exist $N_0 > 0$ and $\alpha > 1$ such that $\frac{A(n)}{A(n+1)} < \frac{1}{\alpha}$ for $n \geq N_0$, so $A(N_0) \cdot \alpha^n \leq A(N_0 + n)$. A simple computation gives us the estimation $|(A(N_0), M) \cap A| \leq \log_\alpha(M)$; hence, there exist $C, D > 0$ such that $|[0,M) \cap A| \leq C\log M + D$. Obviously, the same estimation holds for any element of the ideal $\mathcal{A}$. By [7] there exists a Sidon set A such that $|[1,M] \cap A| > C_1\sqrt[3]{M}$ for some $C_1 > 0$. If this set were an almost thin set then we would have $C_1\sqrt[3]{M} \leq C\log(M) + D$ for any M, which is impossible. □

Let us start with an easy lemma.

Lemma 1. *Suppose that $A \subseteq [1,\infty) \cap \mathbb{N}$ is a set such that*

$$2 \cdot A(n) \leq A(n+1). \tag{1}$$

Then A is a Sidon set.

Proof. Suppose that $A(n) + A(m) = A(k) + A(l)$ for some $n \leq m$ and $k \leq l$. If $l < m$ then we would have $A(k) + A(l) \leq 2A(l) \leq A(l+1) \leq A(m) < A(m) + A(n)$, which is impossible. Hence $m = l$ and therefore $n = k$. □

Theorem 4. *Every $\mathcal{A}$ set is in the ideal $\mathcal{S}i$.*

Proof. It suffices to observe that since any almost thin set satisfies the estimate $A(N_0) \cdot \alpha^n \leq A(N_0 + n)$, we can easily divide it into a finite amount of sets which fulfills (1). □

It is straightforward to see that the Sidon ideal is not a P-ideal. We can use the same example of sets ($A_k = \{n! + k \colon n \in \mathbb{N}\}$) from the Lemma 1.2.8 from [4] since the sets A_k belong to $\mathcal{S}i$. The Figure 1 is a part of the Kowitz Diagram (see [6], page 20) with the new ideal $\mathcal{S}i$ in the appropriate place.

We show that all inclusions in the Figure 1 concerning the Sidon ideal are irreversible. By virtue of Example 1 it suffices to show the following theorem.

$$\begin{tikzcd}
& \mathcal{T} \arrow[d] & \\
& \mathcal{A} \arrow[d] & \\
\mathcal{D}_{\mathit{fin}} & \mathcal{S}i \arrow[l] \arrow[r] \arrow[d] & \mathrm{W} \\
& \mathcal{L}ac &
\end{tikzcd}$$

Figure 1: A part of the Kowitz diagram with the Sidon ideal

Theorem 5. *There exists a set* $A \in \mathrm{W} \cap \mathcal{L}ac \cap \mathcal{D}_{\mathit{fin}}$ *which is not in the ideal* $\mathcal{S}i$.

Definition 14. A set $A \subseteq \mathbb{N}$ is said to be 3-*artihmetic free* if there are no $a, b, c \in A$, where $a < b < c$, such that $2 \cdot b = a + c$.

Definition 15. A set $A \subseteq \mathbb{N}$ is said to be *sum-free* if there are no $a, b \in A$, where $a < b$, such that $a + b \in A$.

Note that this definition differs from the standard definition of a sum-free set in Schur's theorem, because we assume that $a < b$ not $a \leq b$. Notice that if a set $A \subseteq \mathbb{N}$ is 3-arithmetic free then $A \in \mathrm{W}$. Also, if a set $A \subseteq \mathbb{N}$ is sum-free, then $A \in \mathcal{D}_{\mathit{fin}}$ — this is a standard argument which mimics the argument from [3] that a $\mathcal{D}$-sparse set is in the ideal $\mathcal{D}$ (see Proposition 4.3 (1) in [3]). Suppose that n, C, M, K are natural numbers. Define

$$\mathit{Block}(n, C, M, K) = \{M^l + C \cdot 2^k \colon k = 0, 1, \ldots, n-1; l = 1, 2, \ldots, K\}.$$

A simple computation shows that if we assume that $M > C \cdot 2^n$ then $\mathit{Block}(n, C, M, K)$ is both 3-arithmetic and sum-free. Let us formulate the following lemma.

Lemma 2. *For any sequence* (N_k) *of natural numbers and for any sequence* (A_k) *of finite sets, each of which are both* 3*-arithmetic and sum-free sets of natural numbers, we can find a sequence* (m_k) *of natural numbers such that*

1. *the set* $A^* = \bigcup_{k=1}^{\infty} A_k + m_k$ *is both* 3*-arithmetic and sum-free;*
2. *for all* k, $\mathit{dist}(A_k + m_k, A_{k+1} + m_{k+1}) \geq N_k$.

Proof. The proof is a straightforward inductive construction. □

Lemma 3. *The set* $\mathit{Block}(n, C, C \cdot 2^n + 1, (n-1) \cdot \binom{n}{2} + 1)$ *is not a sum of* $n - 1$ *Sidon sets.*

Proof. By way of contradiction, suppose that $Block(n, C, C\cdot 2^n+1, (n-1)\cdot\binom{n}{2}+1) = \bigcup_{j=1}^{n-1} S_j$, where S_j are Sidon sets. For each $l = 1, \ldots (n-1)\cdot\binom{n}{2}+1$ by the pigeonhole principle we can find $k_1(l), k_2(l) \in \{0, \ldots, n-1\}$, $k_1(l) < k_2(l)$, and $j(l) \in \{1, \ldots, n-1\}$ such that $M_l + C\cdot 2^{k_1(l)}, M_l + C\cdot 2^{k_2(l)} \in S_{j(l)}$. Since there are at most $(n-1)\cdot\binom{n}{2}$ triplets $(k_1, k_2, j) \in \{0, \ldots, n-1\}^2 \times \{1, \ldots, n-1\}$, such that $k_1 < k_2$, again by the pigeonhole principle we can find $l_1 < l_2$ such that $(k_1(l_1), k_2(l_1), j(l_1)) = (k_1(l_2), k_2(l_2), j(l_2))$. Let $\hat{k_1} = k_1(l_1) = k_1(l_2)$, $\hat{k_2} = k_2(l_1) = k_2(l_2)$, and $\hat{j} = j(l_1) = j(l_2)$. Then we have $M^{l_1} + C\cdot 2^{\hat{k_1}}, M^{l_1} + C\cdot 2^{\hat{k_2}}, M^{l_2} + C\cdot 2^{\hat{k_1}}, M^{l_2} + C\cdot 2^{\hat{k_2}} \in S_{\hat{j}}$ and this is impossible since $S_{\hat{j}}$ is a Sidon set. □

Proof of Theorem 5. Now apply Lemma 2 to the sequences $N_n = n$ and

$$A_n = Block(n, n, n\cdot 2^n + 1, (n-1)\cdot\binom{n}{2} + 1)$$

and we obtain a suitable sequence (m_n) and define a set

$$A^* = \bigcup_{n=2} Block(n, n, n\cdot 2^n + 1, (n-1)\cdot\binom{n}{2} + 1) + m_n.$$

By Lemma 2 this set is both 3-arithmetic and sum-free. Hence $A^* \in$ W and $A^* \in \mathcal{D}_{fin}$. It is easy to see that also $A^* \in \mathcal{L}ac$. By Lemma 3, A^* is not a union of finitely many Sidon sets; therefore, $A^* \notin \mathcal{S}i$. □

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References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, A dense infinite Sidon sequence, *European J. Combin.* **2** (1) (1981), 1-11.

[2] K. Drakakis, A review of the available construction methods for Golomb rulers, *Adv. Math. Commun.* **3** (3) (2009), 235-250.

[3] R. Filipów, On Hindman spaces and the Bolzano–Weierstrass property, *Topology Appl.* **160** (15) (2013), 2003-2011.

[4] J.Flašková, *Ultrafilters and Small Sets*, Ph.D. thesis, Univerzita Karlova , 2006.

[5] P.M. Kayll, A note on weak Sidon sequences, *Discrete Math.* **299** (1) (2005), 141-144.

[6] K. Kowitz, *The use of Katětov order in the study of topological spaces and ultrafilters*, Ph.D. thesis, University of Gdańsk, 2023.

[7] A.M.Mian and S.Chowla, On the B_2 sequences of Sidon, *Proc. Nat. Acad. Sci. India Sect. A* **14** (1944), 3-4.

[8] I.Z.Ruzsa, An infinite Sidon sequence, *J. Number Theory* **68** (1) (1998), 63-71.