ON THE EXISTENCE OF $\{p, q\}$-DIOPHANTINE QUADRUPLES AND QUINTUPLES

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Abstract

Let S be a set of primes. We call an m-tuple $(a_1, \dots, a_m)$ of distinct positive integers S-Diophantine if, for all $i \neq j$, the integers $s_{i,j} := a_i a_j + 1$ have only prime divisors coming from the set S, i.e., if all $s_{i,j}$ are S-units. In this paper we prove that for $S = \{p, q\}$, no S-Diophantine quadruple exists if $p \equiv 3 \mod 4$ or $q \equiv 3 \mod 4$. We also prove that for $S = \{p, q\}$ no S-Diophantine quintuple exists.

1. Introduction

Let A be a set of k distinct, positive integers. Győry, Sárközy, and Stewart [2] considered the product

$$\Pi = \prod_{\substack{a,b \in A \\ a \neq b}} ab + 1,$$

and found lower bounds for the number ω of prime factors of Π in terms of k. In particular, they showed that $\omega \gg \log k$. They also conjectured that the largest prime factor of $(ab+1)(ac+1)(bc+1)$ tends to infinity as $\max\{a, b, c\} \to \infty$. A weaker form, namely that the largest prime factor of

$$(ab+1)(ac+1)(bd+1)(cd+1)$$

tends to infinity as $\max\{a, b, c, d\} \to \infty$, was proved by Stewart and Tijdeman [5], and the full conjecture was proved independently by Corvaja and Zannier [1] and Hernández and Luca [3]. We want to emphasize that the results of Stewart and Tijdeman [5] are effective, while the results of Corvaja and Zannier [1] and Hernández and Luca [3] are ineffective.

In order to obtain sharp lower bounds for ω for small k we introduce the notion of S-Diophantine m-tuples. Let S be a set of primes. We call an m-tuple $(a_1, \dots, a_m)$

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of distinct, positive integers *S-Diophantine* if, for all $i \neq j$, the integers $s_{i,j} := a_i a_j + 1$ have only prime divisors coming from the set S, i.e., if all $s_{i,j}$ are S-units. We are interested in finding lower bounds for m in terms of $|S|$. Indeed the result of Győry, Sàrközy, and Stewart [2] implies that for $m \geq \exp(C|S|)$, where C is an effectively computable constant, no S-Diophantine m-tuple exists. This bound seems to be far from the truth. However, it seems to be a difficult problem to find sharp bounds for m in terms of $|S|$ even for small values of $|S|$, as for $|S| = 2$ or $|S| = 3$. Let us note that the case that $|S| = 3$ has been studied by the author in [10]. In particular, all $\{p, q, r\}$-Diophantine quadruples with $2 \leq p < q < r \leq 100$ have been found.

Let us note that in the context of S-Diophantine tuples, the results of Corvaja, Zannier, Hernández, and Luca [1, 3] imply that for a fixed, finite set of primes S only finitely many S-Diophantine triples exist. Moreover, the result due to Stewart and Tijdeman [5] implies that for a fixed set of primes S all S-Diophantine quadruples can be effectively computed. However, the following conjecture is still open.

Conjecture 1. Let $S = \{p, q\}$ be a set of primes with $p < q$. Then no S-Diophantine quadruple exists.

Conjecture 1 has been confirmed in the following special cases:

- If $p^2 \nmid q^{\mathrm{ord}_p(q)} - 1$, $q^2 \nmid p^{\mathrm{ord}_q(p)} - 1$, and $q < p^\xi$ holds for some $\xi > 1$, then there exists an effectively computable constant $C = C(\xi)$ such that for all such primes $p, q > C$ no S-Diophantine quadruple exists (see [6]).
- No S-Diophantine quadruple exists, if $p \equiv q \equiv 3 \mod 4$ (see [7]).
- No S-Diophantine quadruple exists, if $p = 2$ or $p = 3$ (see [9]).
- No S-Diophantine quadruple exists, if $p \equiv 3 \mod 4$ and
$$v_q(p^{q-1} - 1) \geq 2, \qquad \text{and} \qquad v_p(q^{p-1} - 1) \geq \max\left\{2, \frac{\log q}{\log p}\right\},$$
where $v_p(x)$ and $v_q(x)$ denote the p-adic and q-adic valuation of x, respectively (see [9]).
- No S-Diophantine quadruple exists, if $p, q < 10^5$ (see [8]).
- No S-Diophantine quadruple exists if $q = p + 2$ is a prime and p is large enough (see [4]).

Let us note that most of the results listed above use results on lower bounds for linear forms in complex and p-adic logarithms. However, in this paper we can prove by using only elementary methods (mainly divisibility properties) the following theorem.

Theorem 1. *Assume that $S = \{p, q\}$ is a set of primes with at least one of p or q congruent to 3 modulo 4. Then there does not exist an S-Diophantine quadruple.*

In other words, to prove Conjecture 1 we only have to consider primes with $p \equiv q \equiv 1 \mod 4$. Although a complete proof of Conjecture 1 seems to be out of reach at the moment, at least we can prove that no S-Diophantine quintuple exists if $|S| = 2$.

Theorem 2. *Let $p < q$ be prime numbers and $S = \{p, q\}$. Then no S-Diophantine quintuple exists.*

In the next section we list some useful results concerning S-Diophantine m-tuples, which will be frequently used in the proofs of Theorems 1 and 2. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2.

2. Preliminaries

Let $S = \{p, q\}$ be a set of two primes with $p < q$ and let (a, b, c, d) be a hypothetical S-Diophantine quadruple. Then we write

$$\begin{aligned} ab + 1 &= p^{\alpha_1} q^{\beta_1}, & bc + 1 &= p^{\alpha_4} q^{\beta_4}, \\ ac + 1 &= p^{\alpha_2} q^{\beta_2}, & bd + 1 &= p^{\alpha_5} q^{\beta_5}, \\ ad + 1 &= p^{\alpha_3} q^{\beta_3}, & cd + 1 &= p^{\alpha_6} q^{\beta_6}. \end{aligned}$$

Moreover, we denote by s_i the S-unit $p^{\alpha_i} q^{\beta_i}$ for $i = 1, \ldots, 6$. If we compute $abcd$ in different ways, we obtain

$$\begin{aligned} abcd = (ab)(cd) &= (s_1 - 1)(s_6 - 1) = s_1 s_6 - s_1 - s_6 + 1 \\ &= (ac)(bd) = (s_2 - 1)(s_5 - 1) = s_2 s_5 - s_2 - s_5 + 1 \\ &= (ad)(bc) = (s_3 - 1)(s_4 - 1) = s_3 s_4 - s_3 - s_4 + 1. \end{aligned}$$

Therefore, we obtain the three non-linear S-unit equations

$$s_1 s_6 - s_1 - s_6 = s_2 s_5 - s_2 - s_5, \tag{1}$$
$$s_3 s_4 - s_3 - s_4 = s_2 s_5 - s_2 - s_5, \tag{2}$$
$$s_1 s_6 - s_1 - s_6 = s_3 s_4 - s_3 - s_4. \tag{3}$$

We start with the following simple divisibility condition which was proved in [6, Lemma 2.1].

Lemma 1 ([6]). *Let (a, b, c) be an S-Diophantine triple, with $a < b < c$, then $s \nmid t$ with $s = ac + 1$ and $t = bc + 1$.*

An immediate consequence of Lemma 1 is that we can exclude the following relations between exponents:

$$\alpha_2 = \alpha_4, \quad \alpha_3 = \alpha_5, \quad \alpha_3 = \alpha_6, \quad \alpha_5 = \alpha_6,$$
$$\beta_2 = \beta_4, \quad \beta_3 = \beta_5, \quad \beta_3 = \beta_6, \quad \beta_5 = \beta_6.$$

On the other hand, we have the following lemma (cf. [6, Proposition 1] or [8, Lemma 2.1]), which one obtains by taking p-adic and q-adic valuations applied to Equations (1), (2), and (3).

Lemma 2 ([6]). *The smallest two exponents occuring in each of the three quadruples* $(\alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $(\alpha_1, \alpha_2, \alpha_5, \alpha_6)$, *and* $(\alpha_1, \alpha_3, \alpha_4, \alpha_6)$ *coincide. The same statement also holds with* α *replaced by* β.

The following lemma also proves to be useful and yields upper bounds for a, b, c, and d. A proof of it can be found in [6, Lemma 3].

Lemma 3 ([6]). *We have*

$$\begin{aligned}
a &| \gcd\left(\frac{s_2 - s_1}{\gcd(s_2, s_1)}, \frac{s_3 - s_1}{\gcd(s_3, s_1)}, \frac{s_3 - s_2}{\gcd(s_3, s_2)}\right),\\
b &| \gcd\left(\frac{s_4 - s_1}{\gcd(s_4, s_1)}, \frac{s_5 - s_1}{\gcd(s_5, s_1)}, \frac{s_5 - s_4}{\gcd(s_5, s_4)}\right),\\
c &| \gcd\left(\frac{s_4 - s_2}{\gcd(s_4, s_2)}, \frac{s_6 - s_2}{\gcd(s_6, s_2)}, \frac{s_6 - s_4}{\gcd(s_6, s_4)}\right),\\
d &| \gcd\left(\frac{s_5 - s_3}{\gcd(s_5, s_3)}, \frac{s_6 - s_3}{\gcd(s_6, s_3)}, \frac{s_6 - s_5}{\gcd(s_6, s_5)}\right).
\end{aligned}$$

We also use the following result which is the content of [7, Section 4].

Lemma 4 ([7]). *Let* p, q *be odd primes and assume that* (a, b, c, d) *is a* $\{p, q\}$*-Diophantine quadruple. Then the following system of equations cannot hold:*

$$\begin{aligned}
ab + 1 &= q^{\beta_1}, & bc + 1 &= p^{\alpha_4} q^{\beta_4},\\
ac + 1 &= p^{\alpha_2}, & bd + 1 &= p^{\alpha_5},\\
ad + 1 &= p^{\alpha_3} q^{\beta_3}, & cd + 1 &= q^{\beta_6}.
\end{aligned} \tag{4}$$

3. Non-Existence of S-Diophantine Quadruples

This section is devoted to the proof of Theorem 1. A key result to establish this theorem is a result obtained by Ziegler [9, Proposition 5.1].

Lemma 5 ([9]). *Assume that* $p \equiv 3 \mod 4$. *Then one of the four cases in Table 1 holds.*

Case	The α exponents	The β exponents
I	$0=\alpha_1=\alpha_6<\alpha_4=\alpha_5<\alpha_2<\alpha_3$	$\beta_1=\beta_2=\beta_3<\beta_4<\beta_5<\beta_6$
II	$0=\alpha_1=\alpha_6<\alpha_2=\alpha_5<\alpha_3,\alpha_4$	$\beta_3=\beta_4<\beta_1=\beta_2<\beta_5<\beta_6$
III	$0=\alpha_2=\alpha_5<\alpha_1=\alpha_3\leq\alpha_4<\alpha_6$	$\beta_1=\beta_6<\beta_3=\beta_4<\beta_2<\beta_5$
IV	$0=\alpha_3=\alpha_4<\alpha_1=\alpha_2<\alpha_5<\alpha_6$	$\beta_1=\beta_6<\beta_2=\beta_5<\beta_3,\beta_4$

Table 1: Restrictions to the exponents

Note that if necessary we can exchange the roles of p and q to ensure that $p\equiv 3 \mod 4$. By the result due to Szalay and the author [6] we may assume that $q\equiv 1 \mod 4$, i.e., $q\geq 5$, and due to the author's result [9] we may assume that $p\neq 3$, i.e., $p\geq 7$. In order to prove Theorem 1 we have to show that the restrictions from Lemma 5, i.e., the cases from Table 1, yield contradictions.

Before we prove Theorem 1 we prove two lemmas first. We start with the following lemma.

Lemma 6. *Assume that Case III of Table 1 holds. Then we have* $d<1.03\cdot q^{2(\beta'-\beta)}$ *and* $b<1.03\cdot q^{\beta'-\beta}$.

Proof. First, we note that due to Lemma 3 we have

$$d\,\left|\,\frac{s_6-s_4}{\gcd(s_4,s_6)}=\frac{p^{\alpha_6}q^{\beta}-p^{\alpha'}q^{\beta'}}{p^{\alpha'}q^{\beta}}=p^{\alpha_6-\alpha'}-q^{\beta'-\beta}\right.$$

and, in particular, we obtain $d<p^{\alpha_6-\alpha'}$.

Moreover, we have

$$\frac{d}{b}=\frac{s_3-1}{s_1-1}=\frac{p^{\alpha}q^{\beta'}-1}{p^{\alpha}q^{\beta}-1}=q^{\beta'-\beta}\cdot\frac{1-p^{-\alpha}q^{-\beta'}}{1-p^{-\alpha}q^{-\beta}}.$$

Since we may assume that $\alpha'>\alpha>0$, $\beta>0$, $p\geq 7$, and $q\geq 5$, we get

$$q^{\beta'-\beta}<\frac{d}{b}<1.03\cdot q^{\beta'-\beta}.$$

Also, note that $s_3^{-1}=p^{-\alpha}q^{-\beta'}<s_1^{-1}=p^{-\alpha}q^{-\beta}$.

Similarly we obtain

$$\frac{d}{b}=\frac{s_6-1}{s_4-1}=\frac{p^{\alpha_6}q^{\beta}-1}{p^{\alpha'}q^{\beta'}-1}=p^{\alpha_6-\alpha'}q^{\beta-\beta'}\cdot\frac{1-p^{-\alpha_6}q^{-\beta}}{1-p^{-\alpha'}q^{-\beta'}}>p^{\alpha_6-\alpha'}q^{\beta-\beta'}.$$

Comparing lower and upper bounds for d/b yields

$$p^{\alpha_6-\alpha'}q^{\beta-\beta'}<\frac{d}{b}<1.03\cdot q^{\beta'-\beta},$$

and hence we get $p^{\alpha_6-\alpha'} < 1.03 \cdot q^{2(\beta'-\beta)}$. Therefore, we obtain

$$d < p^{\alpha_6-\alpha'} < 1.03 \cdot q^{2(\beta'-\beta)}.$$

Now, we obtain the inequality for b by the following simple calculation:

$$b = d \cdot \frac{b}{d} < \frac{1.03 \cdot q^{2(\beta'-\beta)}}{q^{\beta'-\beta}} < 1.03 \cdot q^{\beta'-\beta}.$$

□

Next, we prove the following lemma.

Lemma 7. *Assume that Case IV of Table 1 holds. Then we have* $b < 1.042 \cdot q^{\beta'-\beta}$.

Proof. We use Lemma 3 to obtain

$$d \left| \frac{s_6 - s_5}{\gcd(s_6, s_5)} = \frac{p^{\alpha_6}q^{\beta} - p^{\alpha'}q^{\beta'}}{p^{\alpha'}q^{\beta}} = p^{\alpha_6-\alpha'} - q^{\beta'-\beta}\right. ,$$

which implies $d < p^{\alpha_6-\alpha'}$. Further let us compute

$$\frac{c}{b} = \frac{s_2-1}{s_1-1} = \frac{p^{\alpha}q^{\beta'}-1}{p^{\alpha}q^{\beta}-1} = q^{\beta'-\beta} \cdot \frac{1-p^{-\alpha}q^{-\beta'}}{1-p^{-\alpha}q^{-\beta}} < 1.03 \cdot q^{\beta'-\beta}.$$

The last inequality comes from the fact that we may assume that $\alpha, \beta > 0$ because of Lemma 4. Also, note that we may assume that $p \geq 7$ and $q \geq 5$.

On the other hand we also have

$$\frac{c}{b} = \frac{s_6-1}{s_5-1} = \frac{p^{\alpha_6}q^{\beta}-1}{p^{\alpha'}q^{\beta'}-1} = p^{\alpha_6-\alpha'}q^{\beta-\beta'} \cdot \frac{1-p^{-\alpha_6}q^{-\beta}}{1-p^{-\alpha'}q^{-\beta'}} > p^{\alpha_6-\alpha'}q^{\beta-\beta'}.$$

The last inequality comes from the fact that $s_6^{-1} = p^{-\alpha_6}q^{-\beta} < p^{-\alpha'}q^{-\beta'} = s_5^{-1}$ holds. Combining the inequalities for c/b we get

$$p^{\alpha_6-\alpha'} < 1.03 \cdot q^{2(\beta'-\beta)}.$$

Therefore, we obtain from the previously found upper bound for d the upper bound

$$d < p^{\alpha_6-\alpha'} < 1.03 \cdot q^{2(\beta'-\beta)}.$$

Next, we note that

$$ad = q^{\beta_3} - 1 = q^{\beta_3}(1 - q^{-\beta_3}) \geq 0.992 \cdot q^{\beta_3},$$

since $\beta_3 \geq 3$. Also, note that $q^{\beta_3} = s_3 > s_2 = p^{\alpha}q^{\beta'}$. These two inequalities together with the upper bound for d yields

$$a \geq \frac{0.992 \cdot q^{\beta_3}}{d} > \frac{0.992 \cdot q^{\beta_3+2\beta-2\beta'}}{1.03} > 0.96 \cdot p^{\alpha}q^{2\beta-\beta'}.$$

Hence, we obtain

$$b = \frac{s_1 - 1}{a} < \frac{p^\alpha q^\beta}{0.96 \cdot p^\alpha q^{2\beta - \beta'}} = 1.042 \cdot q^{\beta' - \beta}.$$

□

Now, we start with the proof of Theorem 1.

Proof of Theorem 1. We consider the four cases as shown in Table 1.

`Case I of Table 1.` Due to Lemma 3 we have that

$$d \left| \frac{s_5 - s_3}{\gcd(s_5, s_3)} = \frac{p^{\alpha_5} q^{\beta_5} - p^{\alpha_3} q^{\beta_3}}{p^{\alpha_5} q^{\beta_3}} = q^{\beta_5 - \beta_3} - p^{\alpha_3 - \alpha_5}. \right.$$

This implies $d < q^{\beta_5 - \beta_3}$. Thus we obtain that

$$b + 1 = \frac{s_5 - 1}{d} + 1 > \frac{s_5}{d} = p^{\alpha_5} q^{\beta_3} > q^{\beta_3}.$$

We also have $\beta_3 = \beta_1$ and $\alpha_1 = 0$, which yields

$$b + 1 > q^{\beta_3} = q^{\beta_1} = s_1 = ab + 1 \geq b + 1,$$

which is a contradiction.

`Case II of Table 1.` In this case we get

$$\begin{aligned} \left| \frac{cd+1}{bd+1} \cdot \frac{b}{c} - 1 \right| &= \left| \frac{p^{\alpha_6} q^{\beta_6} \cdot b - p^{\alpha_5} q^{\beta_5} \cdot c}{p^{\alpha_5} q^{\beta_5} \cdot c} \right| = \left| \frac{N}{c p^{\alpha_5}} \right| \\ &= \left| \frac{c - b}{bcd + c} \right| < \frac{1}{bd+1} = \frac{1}{p^{\alpha_5} q^{\beta_5}}, \end{aligned}$$

where N is some integer unequal to 0. Note that $N = 0$ would imply $b = c$, which is excluded. This inequality yields

$$1 \leq |N| < \frac{c}{q^{\beta_5}}$$

and therefore $c > q^{\beta_5}$. By a similar argument, we also obtain a lower bound for b. Namely, we consider

$$\begin{aligned} \left| \frac{bd+1}{ad+1} \cdot \frac{a}{b} - 1 \right| &= \left| \frac{p^{\alpha_5} q^{\beta_5} \cdot a - p^{\alpha_3} q^{\beta_3} \cdot b}{p^{\alpha_3} q^{\beta_3} \cdot b} \right| = \left| \frac{N}{b p^{\alpha_3 - \alpha_5}} \right| \\ &= \left| \frac{b - a}{abd + b} \right| < \frac{1}{ad+1} = \frac{1}{p^{\alpha_3} q^{\beta_3}}, \end{aligned}$$

and deduce that

$$1 \le |N| < \frac{b}{p^{\alpha_5} q^{\beta_3}}.$$

Hence, we have $b > p^{\alpha_5} q^{\beta_3}$. Combining the lower bounds for b and c we obtain

$$p^{\alpha_5} q^{\beta_5} < p^{\alpha_5} q^{\beta_3+\beta_5} < bc < bd + 1 = p^{\alpha_5} q^{\beta_5},$$

which is an obvious contradiction.

`Case III of Table 1.` Let us write $\alpha = \alpha_1 = \alpha_3$, $\alpha' = \alpha_4$, $\beta = \beta_1 = \beta_6$, and $\beta' = \beta_3 = \beta_4$. With this notation we have

$$\begin{aligned} ab + 1 &= s_1 = p^{\alpha} q^{\beta}, & ac + 1 &= s_2 = q^{\beta_2}, & ad + 1 &= s_3 = p^{\alpha} q^{\beta'}, \\ bc + 1 &= s_4 = p^{\alpha'} q^{\beta'}, & bd + 1 &= s_5 = q^{\beta_5}, & cd + 1 &= s_6 = p^{\alpha_6} q^{\beta}. \end{aligned}$$

Let us note that, due to Lemma 4, $\alpha, \beta \ne 0$, that is, we have $0 < \alpha \le \alpha'$ and $0 < \beta < \beta'$.

Next, we note that by Lemma 3 we have that

$$c \left| \frac{s_4 - s_2}{\gcd(s_4, s_2)} = \frac{p^{\alpha'} q^{\beta'} - q^{\beta_2}}{q^{\beta'}} = p^{\alpha'} - q^{\beta_2 - \beta'} \right.,$$

which implies $c < p^{\alpha'}$. Together with the bound for b, which we obtained in Lemma 6, we find

$$p^{\alpha'} q^{\beta'} = bc + 1 < 1.03 \cdot q^{\beta' - \beta} p^{\alpha'} + 1 < p^{\alpha'} q^{\beta'},$$

which is a contradiction.

`Case IV of Table 1.` Let us write $\alpha = \alpha_1 = \alpha_2$, $\alpha' = \alpha_5$, $\beta = \beta_1 = \beta_6$, and $\beta' = \beta_2 = \beta_5$. With this notation we have

$$\begin{aligned} ab + 1 &= s_1 = p^{\alpha} q^{\beta}, & ac + 1 &= s_2 = p^{\alpha} q^{\beta'}, & ad + 1 &= s_3 = q^{\beta_3}, \\ bc + 1 &= s_4 = q^{\beta_4}, & bd + 1 &= s_5 = p^{\alpha'} q^{\beta'}, & cd + 1 &= s_6 = p^{\alpha_6} q^{\beta}. \end{aligned}$$

Again we have $\alpha, \beta \ne 0$ due to Lemma 4, that is, we have $0 < \alpha \le \alpha'$ and $0 < \beta < \beta'$.

Next, let us find an upper bound for c. Indeed we find

$$c \left| \frac{s_4 - s_2}{\gcd(s_4, s_2)} = \frac{q^{\beta_4} - p^{\alpha} q^{\beta'}}{q^{\beta'}} = q^{\beta_4 - \beta'} - p^{\alpha} \right.,$$

which yields $c < q^{\beta_4 - \beta'}$. With this upper bound for c combined with the upper bound for b obtained in Lemma 7 we obtain

$$q^{\beta_4} = bc + 1 < 1.042 \cdot q^{\beta' - \beta} q^{\beta_4 - \beta'} + 1 = 1.042 \cdot q^{\beta_4 - \beta} + 1.$$

This inequality cannot hold unless $\beta = 0$. However, we can exclude the case that $\beta = 0$ due to Lemma 4. □

4. Non-Existence of S-Diophantine Quintuples

This section is devoted to the proof of Theorem 2. Therefore, let (a, b, c, d, e) be a $\{p, q\}$-Diophantine quintuple with $a < b < c < d < e$. Then we write

$$\begin{aligned} ab+1 &= s_1 = p^{\alpha_1}q^{\beta_1}, & cd+1 &= s_6 = p^{\alpha_6}q^{\beta_6},\\ ac+1 &= s_2 = p^{\alpha_2}q^{\beta_2}, & ae+1 &= s_7 = p^{\alpha_7}q^{\beta_7},\\ ad+1 &= s_3 = p^{\alpha_3}q^{\beta_3}, & be+1 &= s_8 = p^{\alpha_8}q^{\beta_8},\\ bc+1 &= s_4 = p^{\alpha_4}q^{\beta_4}, & ce+1 &= s_9 = p^{\alpha_9}q^{\beta_9},\\ bd+1 &= s_5 = p^{\alpha_5}q^{\beta_5}, & de+1 &= s_{10} = p^{\alpha_{10}}q^{\beta_{10}}. \end{aligned}$$

Before we start our proof of Theorem 2 we prove restrictions for the exponents α_i and β_i for an S-Diophantine quadruple (a, b, c, d).

Proposition 1. *Assume that (a, b, c, d) is a $\{p, q\}$-Diophantine quadruple. Then one of the 14 cases listed in Table 2 holds.*

Case	exponents	Case	exponents
I	$\alpha_1=\alpha_2=\alpha_3<\alpha_4,\alpha_6<\alpha_5$ $\beta_1=\beta_4=\beta_5<\beta_2,\beta_6<\beta_3$	VIII	$\alpha_1=\alpha_4=\alpha_5<\alpha_2,\alpha_6<\alpha_3$ $\beta_1=\beta_2=\beta_3<\beta_4,\beta_6<\beta_5$
II	$\alpha_1=\alpha_2=\alpha_3<\alpha_4<\alpha_5<\alpha_6$ $\beta_1=\beta_6<\beta_4=\beta_5<\beta_2<\beta_3$	IX	$\alpha_1=\alpha_6<\alpha_4=\alpha_5<\alpha_2<\alpha_3$ $\beta_1=\beta_2=\beta_3<\beta_4<\beta_5<\beta_6$
III	$\alpha_1=\alpha_6<\alpha_2=\alpha_3<\alpha_4<\alpha_5$ $\beta_1=\beta_4=\beta_5<\beta_2<\beta_3<\beta_6$	X	$\alpha_1=\alpha_4=\alpha_5<\alpha_2<\alpha_3<\alpha_6$ $\beta_1=\beta_6<\beta_2=\beta_3<\beta_4<\beta_5$
IV	$\alpha_2=\alpha_5<\alpha_1=\alpha_3<\alpha_4<\alpha_6$ $\beta_1=\beta_6<\beta_3=\beta_4<\beta_2<\beta_5$	XI	$\alpha_1=\alpha_6<\alpha_3=\alpha_4<\alpha_2<\alpha_5$ $\beta_2=\beta_5<\beta_1=\beta_3<\beta_4<\beta_6$
V	$\alpha_2=\alpha_5<\alpha_1=\alpha_4<\alpha_3<\alpha_6$ $\beta_1=\beta_6<\beta_3=\beta_4<\beta_2<\beta_5$	XII	$\alpha_1=\alpha_6<\alpha_3=\alpha_4<\alpha_2<\alpha_5$ $\beta_2=\beta_5<\beta_1=\beta_4<\beta_3<\beta_6$
VI	$\alpha_1=\alpha_6<\alpha_2=\alpha_5<\alpha_3,\alpha_4$ $\beta_3=\beta_4<\beta_1=\beta_2<\beta_5<\beta_6$	XIII	$\alpha_3=\alpha_4<\alpha_1=\alpha_2<\alpha_5<\alpha_6$ $\beta_1=\beta_6<\beta_2=\beta_5<\beta_3,\beta_4$
VII	$\alpha_2=\alpha_5<\alpha_3=\alpha_4<\alpha_1<\alpha_6$ $\beta_1=\beta_6<\beta_3=\beta_4<\beta_2<\beta_5$	XIV	$\alpha_1=\alpha_6<\alpha_3=\alpha_4<\alpha_2<\alpha_5$ $\beta_2=\beta_5<\beta_3=\beta_4<\beta_1<\beta_6$

Table 2: Restrictions on the exponents for the S-Diophantine quadruple (a, b, c, d)

Let us note that the Cases I-VII of Table 2 are the same cases as the Cases VIII-XIV of Table 2 but with the roles of the α's and β's exchanged. This is easily explained by exchanging the roles of p and q.

Proof of Proposition 1. We consider S-unit Equation (2) in view of Lemma 2 and obtain that one of the following four cases holds:

- $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$,

- $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$,
- $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$,
- $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$.

Note that we have discarded the cases that $\alpha_2 = \alpha_4$ and $\alpha_3 = \alpha_5$, since these would imply that $s_2|s_4$ and $s_3|s_5$, respectively, and therefore contradict the content of Lemma 3. Considering S-unit Equation (3) yields

- $\alpha_1 = \alpha_3 \leq \alpha_4, \alpha_6$,
- $\alpha_1 = \alpha_4 \leq \alpha_3, \alpha_6$,
- $\alpha_1 = \alpha_6 \leq \alpha_3, \alpha_4$,
- $\alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6$,
- $\alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3$.

We now go through the lengthy and tedious process to consider each of the 20 possible combinations of these cases.

`Case 1:` $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$ and $\alpha_1 = \alpha_3 \leq \alpha_4, \alpha_6$. In this case we have

$$\alpha_1 = \alpha_2 = \alpha_3 < \alpha_4, \alpha_6 < \alpha_5.$$

This implies $\beta_1 < \beta_2 < \beta_3$ since $s_1 < s_2 < s_3$. Since $s_2|s_4$ is impossible we also have that $\beta_4 < \beta_2$ and therefore $\beta_4 < \beta_2 < \beta_3$. In view of Lemma 2 we obtain that neither β_2 nor β_3 is minimal in the set $\{\beta_2, \beta_3, \beta_4, \beta_5\}$ and therefore we conclude that $\beta_4 = \beta_5$. A similar consideration applied to the set $\{\beta_1, \beta_6, \beta_2, \beta_5\}$ yields that either $\beta_1 = \beta_5$ or $\beta_1 = \beta_6$ holds. In the case that $\beta_1 = \beta_5$ holds we obtain $\beta_1 = \beta_4 = \beta_5$ and some size comparisons and divisibility restrictions coming from Lemma 3 yields Case I of Table 2. On the other hand, $\beta_1 = \beta_6$ yields Case II of Table 2.

`Case 2:` $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$ and $\alpha_1 = \alpha_4 \leq \alpha_3, \alpha_6$. In this case we obtain

$$\alpha_3 < \alpha_4 \leq \alpha_3$$

an obvious contradiction and we can discard this case.

`Case 3:` $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$ and $\alpha_1 = \alpha_6 \leq \alpha_3, \alpha_4$. This case implies

$$\alpha_1 = \alpha_6 < \alpha_2 = \alpha_3 < \alpha_4, \alpha_5.$$

Note that since $s_3 \nmid s_6$ we cannot have $\alpha_3 = \alpha_6$. The ordering of the α's implies $\beta_4 < \beta_2$ since $s_2 \nmid s_4$, and $\beta_2 < \beta_3$ since $s_2 < s_3$. However, due to Lemma 2, this implies that $\beta_4 = \beta_5 < \beta_2 < \beta_3$. Similarly, we can deduce that $\beta_1 < \beta_6$, and since

$\beta_5 < \beta_2$, we deduce that $\beta_1 = \beta_5$ (Lemma 2), i.e., $\beta_1 = \beta_4 = \beta_5$ and we end up in Case III of Table 2 after taking some further size and divisibility restrictions into account.

`Case 4:` $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$ and $\alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6$. In this case we have $\alpha_2 = \alpha_4$, which yields $s_2 | s_4$. Note that this is a contradiction to Lemma 3.

`Case 5:` $\alpha_2 = \alpha_3 < \alpha_4, \alpha_5$ and $\alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3$. This case implies $\alpha_4 \leq \alpha_3 < \alpha_4$, an obvious contradiction.

`Case 6:` $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$ and $\alpha_1 = \alpha_3 \leq \alpha_4, \alpha_6$. First, we note that

$$\alpha_2 = \alpha_5 < \alpha_1 = \alpha_3$$

holds. Due to a combination of Lemma 3 and size comparisons, we obtain $\beta_4 < \beta_2 < \beta_5$, and hence $\beta_3 = \beta_4$. Therefore, we also have $\beta_1 < \beta_3 = \beta_4$ which yields $\beta_1 = \beta_6$ and, after some more computations, we end up in Case IV of Table 2.

`Case 7:` $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$ and $\alpha_1 = \alpha_4 \leq \alpha_3, \alpha_6$. We can easily verify that $\beta_1 < \beta_4 < \beta_2 < \beta_5$ and therefore Lemma 2 yields that $\beta_1 = \beta_6$ and $\beta_4 = \beta_3$. This yields Case V of Table 2.

`Case 8:` $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$ and $\alpha_1 = \alpha_6 \leq \alpha_3, \alpha_4$. This case yields the inequalities $\beta_2 < \beta_5$ and $\beta_1 < \beta_6$. Therefore, Lemma 2 yields $\beta_1 = \beta_2$. Due to Lemma 3 and the assumptions that $\alpha_5 < \alpha_3$ and $\alpha_2 < \alpha_4$ we obtain $\beta_3 < \beta_5$ and $\beta_4 < \beta_2$, respectively. Therefore, we obtain $\beta_3 = \beta_4$ due to Lemma 2. Altogether, this yields Case VI of Table 2.

`Case 9:` $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$ and $\alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6$. First, we note that we have

$$\alpha_2 = \alpha_5 < \alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6.$$

This implies $\beta_1 < \beta_3$ and $\beta_1 < \beta_4$, because $s_1 < s_3$ and $s_1 < s_4$, respectively. But this also implies $\beta_1 = \beta_6$. Moreover, we have $\beta_4 < \beta_2$ and $\beta_3 < \beta_5$ because of Lemma 1. This implies due to Lemma 2 that $\beta_3 = \beta_4$. Therefore, we have

$$\beta_1 = \beta_6 < \beta_3 = \beta_4 < \beta_2 < \beta_5$$

and we obtain Case VII of Table 2.

`Case 10:` $\alpha_2 = \alpha_5 < \alpha_3, \alpha_4$ and $\alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3$. Now, we have

$$\alpha_2 = \alpha_5 < \alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3.$$

This yields $\beta_1 < \beta_2$ and $\beta_1 < \beta_6$, since $s_1 < s_2$ and $s_1 < s_6$, respectively. Therefore, we have $\beta_1 = \beta_5 < \beta_6$ due to Lemma 2. Altogether we have $\alpha_5 < \alpha_6$ and $\beta_5 < \beta_6$ which implies $s_5 | s_6$, a contradiction to Lemma 3.

`Case 11:` $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$ and $\alpha_1 = \alpha_3 \leq \alpha_4, \alpha_6$. First, we deduce

$$\alpha_1 = \alpha_3 = \alpha_4 < \alpha_2, \alpha_5, \alpha_6$$

and hence $\beta_1 < \beta_3, \beta_4$ and $\beta_1 = \beta_6$. Moreover, due to Lemma 3 we deduce that $\beta_2 < \beta_4$ and $\beta_5 < \beta_3$. Hence, by an application of Lemma 2, we have $\beta_2 = \beta_5$. The equations $\beta_1 = \beta_6$ and $\beta_2 = \beta_5$ imply $\alpha_1 < \alpha_6$ and $\alpha_2 < \alpha_5$. Therefore, we have $\alpha_1 = \alpha_2$ due to Lemma 2. We end up with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ which is impossible since $s_2 \nmid s_4$.

`Case 12:` $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$ and $\alpha_1 = \alpha_4 \leq \alpha_3, \alpha_6$. By the same arguments as in the previous case we also get a contradiction in this case.

`Case 13:` $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$ and $\alpha_1 = \alpha_6 \leq \alpha_3, \alpha_4$. In this case we have

$$\alpha_1 = \alpha_6 < \alpha_3 = \alpha_4 < \alpha_2, \alpha_5.$$

Note that $\alpha_1 = \alpha_6$ implies $\beta_1 < \beta_6$ and Lemma 2 yields the three possibilities $\beta_1 = \beta_3 < \beta_6$, $\beta_1 = \beta_4 \leq \beta_6$, and $\beta_3 = \beta_4 < \beta_6$, which we will consider separately.

Assume that $\beta_1 = \beta_3 < \beta_6$. Since $s_5 \nmid s_6$ we have $\beta_5 < \beta_6$, and since $s_3 \nmid s_5$ we have $\beta_5 < \beta_3 = \beta_1$. Thus the minimum of the set $\{\beta_1, \beta_2, \beta_5, \beta_6\}$ is $\beta_2 = \beta_5$ due to Lemma 2. Thus we will end up in Case XI of Table 2.

Next, we consider the case that $\beta_1 = \beta_4 \leq \beta_6$. Since $s_3 \nmid s_5$ we have $\beta_5 < \beta_3$, and since $s_2 \nmid s_4$ he have $\beta_2 < \beta_4$. Similarly as above we deduce that $\beta_2 = \beta_5$. This yields, after some further size considerations, Case XII of Table 2.

Finally we consider the case that $\beta_3 = \beta_4 < \beta_6$. By the same argument as that previously used we obtain $\beta_2 = \beta_5$ and we end up with Case XIV of Table 2. Note that we have $\beta_2 < \beta_4$ due to Lemma 1, since $\alpha_4 < \alpha_2$.

`Case 14:` $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$ and $\alpha_3 = \alpha_4 < \alpha_1, \alpha_6$. First, we only obtain

$$\alpha_3 = \alpha_4 < \alpha_1, \alpha_2, \alpha_5, \alpha_6.$$

However, we still conclude $\beta_5 < \beta_3$ since $s_3 \nmid s_5$, and $\beta_2 < \beta_4$ since $s_2 \nmid s_4$. This implies $\beta_2 = \beta_5$ and therefore we have $\alpha_2 < \alpha_5$. Now, we consider the S-unit Equation (1) and we obtain that one of the cases

- $\alpha_1 = \alpha_2 \leq \alpha_6$,
- $\alpha_1 = \alpha_6 \leq \alpha_2$, or
- $\alpha_2 = \alpha_6 \leq \alpha_1$

holds. Let us note that the case $\alpha_1 = \alpha_6 \leq \alpha_2$ has already been considered before. Therefore, we have to consider the other two cases.

First, let us assume that $\alpha_1 = \alpha_2$. We already know that

$$\alpha_3 = \alpha_4 < \alpha_1 = \alpha_2 \leq \alpha_5, \alpha_6$$

and $\beta_2 = \beta_5 < \beta_3, \beta_4$. Since $s_1 < s_2$ we have $\beta_1 < \beta_2 = \beta_5$ and therefore $\beta_1 = \beta_6 < \beta_2 = \beta_5$ and we end up in Case XIII of Table 2.

If we assume that $\alpha_2 = \alpha_6$, then this implies $\alpha_2 = \alpha_6 \leq \alpha_1, \alpha_5$, and since $s_1 < s_2$ and $s_5 \nmid s_6$ we deduce that $\beta_1 < \beta_2 = \beta_5 < \beta_6$. Therefore, the set $\{\beta_1, \beta_2, \beta_5, \beta_6\}$ has a sole minimum, which contradicts Lemma 2.

`Case 15:` $\alpha_3 = \alpha_4 < \alpha_2, \alpha_5$ and $\alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3$. This case implies $\alpha_3 = \alpha_6$ and therefore we have $s_3 | s_6$. This contradicts Lemma 1 and we can discard this case.

`Case 16:` $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$ and $\alpha_1 = \alpha_3 \leq \alpha_4, \alpha_6$. We have

$$\alpha_4 = \alpha_5 < \alpha_1 = \alpha_3 \leq \alpha_2, \alpha_6.$$

However, this yields $\beta_2 < \beta_3$ and $\beta_4 < \beta_5$ since $s_2 < s_3$ and $s_4 < s_5$, respectively. Moreover, we have $\beta_2 < \beta_4$ by Lemma 1. This implies that the set $\{\beta_2, \beta_3, \beta_4, \beta_5\}$ has a sole minimum contradicting Lemma 2.

`Case 17:` $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$ and $\alpha_1 = \alpha_4 \leq \alpha_3, \alpha_6$. In this case we obtain

$$\alpha_1 = \alpha_4 = \alpha_5 < \alpha_2, \alpha_3, \alpha_6.$$

This implies $\beta_1 < \beta_4 < \beta_5$, and moreover we have $\beta_2 < \beta_4$ and $\beta_3 < \beta_5$, since $s_2 \nmid s_4$ and $s_3 \nmid s_5$. In view of Lemma 2 this implies $\beta_2 = \beta_3$. We also note that β_5 cannot be the minimum of $\{\beta_1, \beta_2, \beta_5, \beta_6\}$. Hence, either $\beta_1 = \beta_2$ or $\beta_1 = \beta_6$ or $\beta_2 = \beta_6$ is the minimum of $\{\beta_1, \beta_2, \beta_5, \beta_6\}$.

If $\beta_1 = \beta_2$ is the minimum, then we have $\beta_1 = \beta_2 = \beta_3$, and some more size and divisibility considerations yield Case VIII of Table 2. If $\beta_1 = \beta_6$ is the minimum, then we have $\beta_1 = \beta_6 < \beta_2 = \beta_3$ and we will end up in Case X of Table 2. If $\beta_2 = \beta_6$ is the minimum, then we get $\beta_2 = \beta_3 = \beta_6$ a contradiction to $s_3 \nmid s_6$.

`Case 18:` $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$ and $\alpha_1 = \alpha_6 \leq \alpha_3, \alpha_4$. We get

$$\alpha_1 = \alpha_6 < \alpha_4 = \alpha_5 < \alpha_2, \alpha_3.$$

Since $s_3 \nmid s_5$ and $s_2 \nmid s_4$ we have $\beta_3 < \beta_5$ and $\beta_2 < \beta_4$, which results in $\beta_2 = \beta_3 < \beta_5$. That is β_5 is not the minimum in $\{\beta_1, \beta_2, \beta_5, \beta_6\}$. Therefore, we have either $\beta_1 = \beta_2$ or $\beta_1 = \beta_6$ or $\beta_2 = \beta_6$.

Let us note that $\beta_1 = \beta_6$ in combination with $\alpha_1 = \alpha_6$ yields $s_1 = s_6$, a contradiction. Also, the case that $\beta_2 = \beta_6$ yields $\beta_2 = \beta_3 = \beta_6$, which contradicts the fact that $s_3 \nmid s_6$. Finally, we note that the case that $\beta_1 = \beta_2$ results in $\beta_1 = \beta_2 = \beta_3$, and some more size and divisibility considerations yield Case IX of Table 2.

`Case 19:` $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$ and $\alpha_3 = \alpha_4 \leq \alpha_1, \alpha_6$. In this case we have $\alpha_3 = \alpha_5$, which contradicts $s_3 \nmid s_5$.

`Case 20:` $\alpha_4 = \alpha_5 < \alpha_2, \alpha_3$ and $\alpha_4 = \alpha_6 \leq \alpha_1, \alpha_3$. In this final case we get $\alpha_5 = \alpha_6$, which contradicts $s_5 \nmid s_6$. □

We start now with the proof of Theorem 2.

Proof of Theorem 2. Let us assume that (a, b, c, d, e) is a $\{p, q\}$-Diophantine quintuple. Then (b, c, d, e) is a $\{p, q\}$-Diophantine quadruple. Therefore, we obtain the same restrictions on the exponents as listed in Table 2, but where the indices are replaced as indicated below:

$$1 \to 4, \qquad 3 \to 8, \qquad 5 \to 9,$$
$$2 \to 5, \qquad 4 \to 6, \qquad 6 \to 10.$$

Thus we obtain that at least one of the restrictions listed in Table 3 holds.

Case	exponents	Case	exponents
I'	$\alpha_4 = \alpha_5 = \alpha_8 < \alpha_6, \alpha_{10} < \alpha_9$ $\beta_4 = \beta_6 = \beta_9 < \beta_5, \beta_{10} < \beta_8$	VIII'	$\alpha_4 = \alpha_6 = \alpha_9 < \alpha_5, \alpha_{10} < \alpha_8$ $\beta_4 = \beta_5 = \beta_8 < \beta_6, \beta_{10} < \beta_9$
II'	$\alpha_4 = \alpha_5 = \alpha_8 < \alpha_6 < \alpha_9 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_6 = \beta_9 < \beta_5 < \beta_8$	IX'	$\alpha_4 = \alpha_{10} < \alpha_6 = \alpha_9 < \alpha_5 < \alpha_8$ $\beta_4 = \beta_5 = \beta_8 < \beta_6 < \beta_9 < \beta_{10}$
III'	$\alpha_4 = \alpha_{10} < \alpha_5 = \alpha_8 < \alpha_6 < \alpha_9$ $\beta_4 = \beta_6 = \beta_9 < \beta_5 < \beta_8 < \beta_{10}$	X'	$\alpha_4 = \alpha_6 = \alpha_9 < \alpha_5 < \alpha_8 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_5 = \beta_8 < \beta_6 < \beta_9$
IV'	$\alpha_5 = \alpha_9 < \alpha_4 = \alpha_8 < \alpha_6 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_8 = \beta_6 < \beta_5 < \beta_9$	XI'	$\alpha_4 = \alpha_{10} < \alpha_8 = \alpha_6 < \alpha_5 < \alpha_9$ $\beta_5 = \beta_9 < \beta_4 = \beta_8 < \beta_6 < \beta_{10}$
V'	$\alpha_5 = \alpha_9 < \alpha_4 = \alpha_6 < \alpha_8 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_8 = \beta_6 < \beta_5 < \beta_9$	XII'	$\alpha_4 = \alpha_{10} < \alpha_8 = \alpha_6 < \alpha_5 < \alpha_9$ $\beta_5 = \beta_9 < \beta_4 = \beta_6 < \beta_8 < \beta_{10}$
VI'	$\alpha_4 = \alpha_{10} < \alpha_5 = \alpha_9 < \alpha_8, \alpha_6$ $\beta_8 = \beta_6 < \beta_4 = \beta_5 < \beta_9 < \beta_{10}$	XIII'	$\alpha_8 = \alpha_6 < \alpha_4 = \alpha_5 < \alpha_9 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_5 = \beta_9 < \beta_8, \beta_6$
VII'	$\alpha_5 = \alpha_9 < \alpha_8 = \alpha_6 < \alpha_4 < \alpha_{10}$ $\beta_4 = \beta_{10} < \beta_8 = \beta_6 < \beta_5 < \beta_9$	XIV'	$\alpha_4 = \alpha_{10} < \alpha_8 = \alpha_6 < \alpha_5 < \alpha_9$ $\beta_5 = \beta_9 < \beta_8 = \beta_6 < \beta_4 < \beta_{10}$

Table 3: Restrictions on the exponents for the S-Diophantine quadruple (b, c, d, e)

By exchanging the roles of p and q we only have to match the Cases I to VII of Table 2 with the Cases I' to XIV' of Table 3. Let us note that the indices $4, 5,$ and 6 appear in both tables. Considering all possible combinations of restrictions, the only instances where the orders of $\alpha_4, \alpha_5, \alpha_6$ and $\beta_4, \beta_5, \beta_6$ coincide are the combinations of the Cases II and VI', I and VIII', I and IX', respectively. That is, if (a, b, c, d, e) is an S-Diophantine quintuple, then we may assume (after possibly exchanging the roles of p and q) that one of the following three cases holds.

`Case 1:` We combine Case II of Table 2 with Case VI' of Table 3 and obtain

$$\alpha_1 = \alpha_2 = \alpha_3 < \alpha_4 = \alpha_{10} < \alpha_5 = \alpha_9 < \alpha_6, \alpha_8,$$
$$\beta_1 = \beta_6 = \beta_8 < \beta_4 = \beta_5 < \begin{cases} \beta_2 < \beta_3, \\ \beta_9 < \beta_{10}. \end{cases}$$

`Case 2:` We combine Case I of Table 2 with Case VIII' of Table 3 and obtain

$$\alpha_1 = \alpha_2 = \alpha_3 < \alpha_4 = \alpha_6 = \alpha_9 < \alpha_5, \alpha_{10} < \alpha_8,$$
$$\beta_1 = \beta_4 = \beta_5 = \beta_8 < \begin{cases} \beta_2, \beta_6 < \beta_3, \\ \beta_6, \beta_{10} < \beta_9. \end{cases}$$

`Case 3:` We combine Case I of Table 2 with Case IX' of Table 3 and obtain

$$\alpha_1 = \alpha_2 = \alpha_3 < \alpha_4 = \alpha_{10} < \alpha_6 = \alpha_9 < \alpha_5 < \alpha_8,$$
$$\beta_1 = \beta_4 = \beta_5 = \beta_8 < \begin{cases} \beta_2, \beta_6 < \beta_3, \\ \beta_6 < \beta_9 < \beta_{10}. \end{cases}$$

Finally, we consider the quadruple (a, c, d, e). That is, by changing the indices according to

$$1 \to 2, \qquad 3 \to 7, \qquad 5 \to 9,$$
$$2 \to 3, \qquad 4 \to 6, \qquad 6 \to 10,$$

due to Proposition 1 we obtain that at least one of the restrictions listed in Table 4 holds.

Case	exponents	Case	exponents
I"	$\alpha_2 = \alpha_3 = \alpha_7 < \alpha_6, \alpha_{10} < \alpha_9$ $\beta_2 = \beta_6 = \beta_9 < \beta_3, \beta_{10} < \beta_7$	VIII"	$\alpha_2 = \alpha_6 = \alpha_9 < \alpha_3, \alpha_{10} < \alpha_7$ $\beta_2 = \beta_3 = \beta_7 < \beta_6, \beta_{10} < \beta_9$
II"	$\alpha_2 = \alpha_3 = \alpha_7 < \alpha_6 < \alpha_9 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_6 = \beta_9 < \beta_3 < \beta_7$	IX"	$\alpha_2 = \alpha_{10} < \alpha_6 = \alpha_9 < \alpha_3 < \alpha_7$ $\beta_2 = \beta_3 = \beta_7 < \beta_6 < \beta_9 < \beta_{10}$
III"	$\alpha_2 = \alpha_{10} < \alpha_3 = \alpha_7 < \alpha_6 < \alpha_9$ $\beta_2 = \beta_6 = \beta_9 < \beta_3 < \beta_7 < \beta_{10}$	X"	$\alpha_2 = \alpha_6 = \alpha_9 < \alpha_3 < \alpha_7 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_3 = \beta_7 < \beta_6 < \beta_9$
IV"	$\alpha_3 = \alpha_9 < \alpha_2 = \alpha_7 < \alpha_6 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_7 = \beta_6 < \beta_3 < \beta_9$	XI"	$\alpha_2 = \alpha_{10} < \alpha_7 = \alpha_6 < \alpha_3 < \alpha_9$ $\beta_3 = \beta_9 < \beta_2 = \beta_7 < \beta_6 < \beta_{10}$
V"	$\alpha_3 = \alpha_9 < \alpha_2 = \alpha_6 < \alpha_7 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_7 = \beta_6 < \beta_3 < \beta_9$	XII"	$\alpha_2 = \alpha_{10} < \alpha_7 = \alpha_6 < \alpha_3 < \alpha_9$ $\beta_3 = \beta_9 < \beta_2 = \beta_6 < \beta_7 < \beta_{10}$
VI"	$\alpha_2 = \alpha_{10} < \alpha_3 = \alpha_9 < \alpha_7, \alpha_6$ $\beta_7 = \beta_6 < \beta_2 = \beta_3 < \beta_9 < \beta_{10}$	XIII"	$\alpha_7 = \alpha_6 < \alpha_2 = \alpha_3 < \alpha_9 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_3 = \beta_9 < \beta_7, \beta_6$
VII"	$\alpha_3 = \alpha_9 < \alpha_7 = \alpha_6 < \alpha_2 < \alpha_{10}$ $\beta_2 = \beta_{10} < \beta_7 = \beta_6 < \beta_3 < \beta_9$	XIV"	$\alpha_2 = \alpha_{10} < \alpha_7 = \alpha_6 < \alpha_3 < \alpha_9$ $\beta_3 = \beta_9 < \beta_7 = \beta_6 < \beta_2 < \beta_{10}$

Table 4: Restrictions on the exponents for the S-Diophantine quadruple (a, c, d, e)

Since all three Cases 1, 2, and 3 imply that $\alpha_2 = \alpha_3$ only the Cases I", II", and XIII" of Table 4 may hold. However, the Cases I", II", and XIII" of Table 4 imply that $\alpha_6 < \alpha_9$ which contradicts the Cases 1, 2, and 3. Thus we have proved that any combination of restrictions leads to a contradiction, thus no S-Diophantine quintuple exists, which finally proves Theorem 2. $\square$

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