ON THE ADDITIVE UNIQUENESS OF GENERALIZED NONAGONAL NUMBERS FOR MULTIPLICATIVE FUNCTIONS

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Abstract

We prove that the set $\mathcal{N}$ of all nonzero generalized nonagonal numbers is an additive uniqueness set. If a multiplicative function f satisfies the equation

$$f(a+b) = f(a) + f(b)$$

for all $a, b \in \mathcal{N}$, then f is the identity function.

1. Introduction

In 1992, C. Spiro [11] introduced the notion of an *additive uniqueness set*, briefly AU set, $E \subset \mathbb{N}$ of a subset S of arithmetic functions, which means $f \in S$ is uniquely determined by the condition $f(a+b) = f(a) + f(b)$ for all $a, b \in E$. She showed that the set of primes is an additive uniqueness set for the set

$$S = \{f \mid f \text{ is multiplicative and } f(p_0) \neq 0 \text{ for some prime } p_0\},$$

where f is multiplicative if $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all a and b with $\gcd(a, b) = 1$. Since her paper was published, many mathematicians have been studying the k-additive uniqueness, briefly k-AU, of various sets of natural numbers with the condition

$$f(a_1 + a_2 + \cdots + a_k) = f(a_1) + f(a_2) + \cdots + f(a_k).$$

In 1999, Chung and Phong [2] showed that the set of triangular numbers is an AU set for the set of multiplicative functions. This set is also a k-AU set with $k \geq 3$

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[9]. However, the set of squares is not a 2-AU set for multiplicative functions [1] and it is a k-AU set for $k \geq 3$ [8].

Let us consider the additive uniqueness of polygonal numbers for multiplicative functions to generalize the above results. The author and colleagues [6] proved that the set $\mathcal{P} = \{\frac{n(3n-2)}{2} \,|\, n \in \mathbb{Z}, n \neq 0\}$ of generalized pentagonal numbers is an AU set for multiplicative functions. This is also a k-AU set for $k \geq 3$ [3, 10].

It is much more difficult to consider the set $\mathcal{P}^+ = \{\frac{n(3n-2)}{2} \,|\, n \in \mathbb{Z}, n \geq 1\}$ of ordinary pentagonal numbers. This set is also a 2-AU set, which was proved in [7]. The set $\mathcal{H}^+ = \{n(2n-1) \,|\, n \in \mathbb{Z}, n \geq 1\}$ of ordinary hexagonal numbers is also a 2-AU set [7]. In [5], it is proved that $\mathcal{H}^+$ is also a k-AU set for all $k \geq 3$. Recently, Hasanalizade and the author [4] showed that the set $\mathcal{O} = \{n(n-2) \,|\, n \in \mathbb{Z}, n \neq 0\}$ of generalized octagonal numbers is not a 2-AU set and is not a 3-AU set, but it is a k-AU set for $k \geq 4$.

In this article we prove the additive uniqueness of the set

$$\begin{aligned}\mathcal{N} &= \left\{ N_n = \frac{n(7n-5)}{2} \;\middle|\; n \in \mathbb{Z}, n \neq 0 \right\} \\ &= \{1, 6, 9, 19, 24, 39, 46, 66, 75, 100, 111, 141, 154, \dots\}\end{aligned}$$

of generalized nonagonal numbers. The following theorem holds.

Theorem 1. *If a multiplicative function f satisfies*

$$f(a+b) = f(a) + f(b)$$

for all nonzero generalized nonagonal numbers a and b, then f is the identity function.

2. Strategy

We use induction to prove the main theorem. That is, assuming $f(n) = n$ for all $n < N$, we show that $f(N) = N$. If $N = ab$ with $a, b \geq 2$ and $\gcd(a, b) = 1$, then $f(N) = f(a)\, f(b) = ab = N$ by the induction hypothesis. So we may check whether $f(p^r) = p^r$ or not for primes p.

In proving we use $f(N_a + N_b) = f(N_a) + f(N_b)$ for suitable a and b. Since two factors n and $7n-5$ of $2N_n = n(7n-5)$ can have a common divisor 5, we cannot split $f(N_n)$ into $f(n)\, f(7n-5)$ for the case.

The proof is done in a few steps. First, we evaluate $f(n)$ for some n's in Lemma 1. Using this evaluation, in Sections 3-5, we prove that $f(p^r) = p^r$ for $p = 3, 5, 7$. In Section 6 we prove that $f(2^r) = 2^r$. Finally, in Section 7, we prove that $f(p^r) = p^r$ for other primes p.

For convenience, we introduce a notation for relatively prime factors. If $n = ab$ and $\gcd(a, b) = 1$, then we write $n = a \times b$. For example, $n = 2 \cdot 2 \cdot 3 \cdot 5$ can be written as

$$4 \times 3 \cdot 5 = 4 \cdot 3 \times 5 = 3 \times 4 \cdot 5,$$

which means

$$\gcd(4, 3 \cdot 5) = \gcd(4 \cdot 3, 5) = \gcd(3, 4 \cdot 5) = 1.$$

Lemma 1. *$f(n) = n$ for $n \leq 11$.*

Proof. We have $f(2) = 2$. Note that

$$f(3)\, f(4) = f(2 \cdot 3 + 2 \cdot 3) = f(2)\, f(3) + f(2)\, f(3) = 4f(3).$$

If $f(3) = 0$, then we cannot determine $f(4)$ yet. In this case, we have $f(9) = 0$ from

$$f(3)\, f(5) = f(9) + f(2)\, f(3)$$

and $f(5) = \frac{1}{2}$ from $f(2)\, f(5) = f(1) + f(9)$.

Also, $f(7) = f(1) + f(2)\, f(3) = 1$ and, thus, $f(4) = f(19)$ by $f(4)\, f(7) = f(9) + f(19)$. Then $f(4) = f(19) = -2$ from

$$f(4)\, f(5) = f(1) + f(19)$$

and $f(23) = \frac{1}{2}$ from

$$f(2)\, f(5)\, f(7) = f(3)\, f(8) + f(2)\, f(23).$$

A contradiction occurs in solving

$$f(4)\, f(23) = f(2)\, f(23) + f(2)\, f(23)$$

and we can conclude that $f(3) \neq 0$.

Then $f(4) = 4$ from $f(3)\, f(4) = f(2)\, f(3) + f(2)\, f(3)$. Note that

$$\begin{aligned}
f(7) = f(2)\, f(3) + f(1) &\iff f(7) = 2f(3) + 1 \\
f(2)\, f(5) = f(9) + f(1) &\iff f(9) = 2f(5) - 1 \\
f(4)\, f(5) = f(19) + f(1) &\iff f(19) = 4f(5) - 1 \\
f(4)\, f(7) = f(19) + f(9) &\iff f(19) = 4f(7) - f(9).
\end{aligned}$$

We obtain

$$4f(5) - 1 = 4(2f(3) + 1) - (2f(5) - 1)$$

from the last two equations and thus

$$3f(5) = 4f(3) + 3.$$

Then, since

$$f(3)\,f(5) = f(9) + f(2)\,f(3) = 2f(5) - 1 + 2f(3),$$

we can find two solutions:

$$\begin{array}{llllll} f(3)=3, & f(5)=5, & f(7)=7, & f(9)=9, & f(19)=19; \\ f(3)=-\frac{1}{4}, & f(5)=\frac{2}{3}, & f(7)=\frac{1}{2}, & f(9)=\frac{1}{3}, & f(19)=\frac{5}{3}. \end{array}$$

Next, we deduce that $f(8) = 8$ or $f(8) = -\frac{2}{3}$ from

$$f(2)\,f(3)\,f(5) = f(3)\,f(8) + f(2)\,f(3).$$

The second solution set cannot satisfy

$$\begin{aligned} f(3)\,f(11) &= f(3)\,f(8) + f(9) \\ f(8)\,f(9) &= f(2)\,f(3)\,f(11) + f(2)\,f(3). \end{aligned}$$

So, $f(n)$ is determined to be n up to 11. □

3. Proof for $f(3^r) = 3^r$

The basic idea of the proof is to use induction under the assumption $f(n) = n$ for all $n < 3^r$. This assumption is too loose. In Section 7 we use $f(3^r) = 3^r$ to prove $f(p^s) = p^s$ for all primes $p > 7$. In this case we need to check $f(3^r)$ for some $3^r > p^s$ with the induction hypothesis that $f(n) = n$ for all $n < p^r$. To do this we need the stronger induction hypothesis for $f(3^r)$. That is, we should find a function $\alpha(x)$ such that $f(n) = n$ for all $n < \alpha(3^r) < p^r$.

Theorem 2. *Let* $\alpha(x) = \frac{14}{17} \cdot \frac{2}{3} \cdot x - \frac{12}{17}$. *If* $f(n) = n$ *for all* $n < \alpha(3^r)$, *then* $f(3^r) = 3^r$

Proof. The weird coefficients of $\alpha(x)$ are determined by the extremal inequality of Case I of Section 7.

If $r = 2s$, then we can show easily $f(3^{2s}) = 3^{2s}$ by using

$$N_{-m} + N_m = m \times \frac{7m+5}{2} + m \times \frac{7m-5}{2} = 7 \times m^2$$

with $m = 3^s$. Note the $a \times b$ notation and that $f(3^2) = 3^2$ was already showed in Lemma 1 and

$$\max\left\{m, \frac{7m+5}{2}, \frac{7m-5}{2}\right\} = \frac{7m+5}{2} < \alpha(m^2) = \frac{14}{17} \cdot \frac{2}{3} m^2 - \frac{12}{17}$$

when $m \geq 9$.

Now assume that $r = 2s+1$. We have that $3^{2s+1} \equiv 3, 13, 5 \pmod{14}$. We consider it in three cases.

Case I: $3^{2s+1} \equiv 3 \cdot 1 \pmod{14}$. We let $3^{2s+1} = 3(14m+1)$ and use

$$\begin{aligned} N_{4m+1} + N_{-10m} &= (4m+1)(14m+1) + 25m(14m+1) \\ &= (14m+1)(29m+1). \end{aligned}$$

Note that $m \equiv 1 \pmod{3}$ and, in the first term $(4m+1)(14m+1)$,

$$\gcd(4m+1, 14m+1) = \gcd(5, m-1) = 1 \text{ or } 5.$$

Since $14m+1 = 3^{2s}$, we have that $m-1$ is not divisible by 5 and $N_{4m+1} = (4m+1) \times (14m+1)$ by the $a \times b$ notation. Similarly, $N_{-10m} = 25m \times (14m+1)$.

For the second term $25m(14m+1)$, if $m = 5^k$, then $14 \cdot 5^k + 1 = 3^{2s}$. However, this is impossible by comparing both sides modulo 8. Thus, m has a divisor d such that $d \neq 1, 5$ and $\gcd(m/d, d) = 1$.

Now consider $(14m+1)(29m+1)$. Since $3 \mid (m-1)$, $5 \nmid (m-1)$ and

$$\gcd(14m+1, 29m+1) = \gcd(14m+1, m-1) = \gcd(15, m-1),$$

we have that $\gcd(14m+1, 29m+1) = 3$. Thus,

$$f(N_{4m+1}) + f(N_{-10m}) = f(4m+1)\, f(14m+1) + f\left(25 \cdot \frac{m}{d}\right) f(d)\, f(14m+1)$$

and

$$\begin{aligned} f(N_{4m+1} + N_{-10m}) &= f\left(3(14m+1) \times \frac{29m+1}{3}\right) \\ &= f(3^{2s+1})\, f\left(\frac{29m+1}{3}\right). \end{aligned}$$

Then, since

$$\begin{aligned} &\max\left\{4m+1, 14m+1, 25 \cdot \frac{m}{d}, d, \frac{29m+1}{3}\right\} \\ &\quad = 14m+1 \\ &\quad < \alpha(3(14m+1)) = \frac{14}{17} \cdot \frac{2}{3} \cdot 3(14m+1) - \frac{12}{17}, \end{aligned}$$

we can deduce that $f(3^{2s+1}) = 3^{2s+1}$ by the induction hypothesis.

Case II: $3^{2s+1} \equiv 3 \cdot 9 \pmod{14}$. For this case, $3^{2s+1} = 3(14m+9)$ and we use

$$\begin{aligned} N_{3m+2} + N_{-12m-7} &= \frac{3}{2}(3m+2)(7m+3) + 3(12m+7)(14m+9) \\ &= \frac{9}{2}(7m+4)(17m+11). \end{aligned}$$

Note that m is a multiple of 9, since $14m+9 = 3^{2s}$. Also, comparing both sides of $14m+9 = 3^{2s}$ modulo 8, we have that m is a multiple of 4. Similarly, we deduce that $\gcd(3m+2, 7m+3) = \gcd(5, m-1) = 1$ by comparing both sides of $14m+9 = 3^{2s}$ modulo 5. Thus, the first term can be written as $3^2 \times \frac{3m+2}{2} \times \frac{7m+3}{3}$. Since $14m+9 = 3^{2s}$, we have that $\gcd(12m+7, 14m+9) = 1$ and thus the second term can be written as $(12m+7) \times 3(14m+9)$.

Since the last term can be written as $9 \times \frac{7m+4}{2} \times (17m+11)$,

$$\begin{aligned}
&\max\left\{9, \frac{3m+2}{2}, \frac{7m+3}{3}, 12m+7, \frac{7m+4}{2}, 17m+11\right\} \\
&= 17m+11 \\
&< \alpha(3(14m+9)) = \frac{14}{17} \cdot \frac{2}{3} \cdot 3(14m+9) - \frac{12}{17}
\end{aligned}$$

and $f(3^{2s+1}) = 3^{2s+1}$ by the induction.

`Case III:` $3^{2s+1} \equiv 3 \cdot 11 \pmod{14}$. In this case, $2s+1 \geq 5$ and we consider $3^{2s+1} = 3^3(14m+9)$ instead of the form $3(14m+11)$. Note that

$$\begin{aligned}
N_{-6m-3} + N_{12m+8} &= 3(2m+1)(21m+13) + 6(3m+2)(28m+17) \\
&= 9(5m+3)(14m+9)
\end{aligned}$$

and

$$\begin{aligned}
\gcd(2m+1, 21m+13) &= \gcd(5, m+3), \\
\gcd(3m+2, 28m+17) &= \gcd(5, m-1), \\
\gcd(5m+3, 14m+9) &= \gcd(m, 3) = 3.
\end{aligned}$$

If $m+3 \equiv 0 \pmod 5$, then $14m+9 \equiv 2 \pmod 5$. We have $3^{2(s-1)} \equiv 2 \pmod 5$ has no solution. So, $\gcd(2m+1, 21m+13) = 1$. By the similar reasoning $\gcd(3m+2, 28m+17) = 1$, too. Thus, we can write, by the $a \times b$ notation,

$$\begin{aligned}
N_{-6m-3} &= 3 \times (2m+1) \times (21m+13) \\
N_{12m+8} &= 3 \times 2(3m+2) \times (28m+17) \\
N_{-6m-3} + N_{12m+8} &= \frac{5m+3}{3} \times 27(14m+9) = \frac{5m+3}{3} \times 3^{2s+1}.
\end{aligned}$$

Then, since the maximal factor $28m+17$ is smaller than $\alpha(3^3(14m+9))$, we can conclude that $f(3^{2s+1}) = 3^{2s+1}$ by the induction hypothesis. □

4. Proof for $f(5^r) = 5^r$

In the previous section we proved that $f(3^r) = 3^r$ by induction under the stronger assumption that $f(n) = n$ for all $n < \alpha(3^r) < 3^r$. We need the similar condition

for $f(5^r)$. That is, the assumption $f(n) = n$ for all $n < 5^r$ is too loose to be used in proving $f(p^s) = p^s$ for other prime $p > 7$ and $p = 2$.

Theorem 3. *Let $\beta(x) = \frac{1}{4}x$. If $f(n) = n$ for all $n \le \beta(5^r)$ with $r \ge 3$, then $f(5^r) = 5^r$.*

Proof. The coefficient of $\beta(x)$ can be reduced to $\frac{17}{70}$ by the extremal inequality of Case I-ii of this proof. For convenience, it is enough to set $\beta(x) = \frac{1}{4}x$.

When r is even, we set $r = 2s$ and use

$$N_{-m} + N_m = 5m \times \frac{7m+5}{2\cdot 5} + 5m \times \frac{7m-5}{2\cdot 5} = 7 \times m^2$$

with $m = 5^s$. Then $f(5^{2s}) = 5^{2s}$ by the induction hypothesis, since the maximal factor $5m = 5\sqrt{5^r}$ is smaller than $\beta(m^2)$.

Now assume that r is odd. Then

$$5^r = 5^{2s+1} \equiv 5, 13, 17 \pmod{28} \quad \text{and} \quad 5^{r-1} = 5^{2s} \equiv 1, 9, 25 \pmod{28}.$$

We check $f(5^r) = f(5 \cdot (28m + t))$ with $t \in \{1, 9, 25\}$.
`Case I:` $5^{2s+1} = 5(28m + 1) \equiv 5 \pmod{28}$. In this case we have $m \equiv 8 \pmod{25}$. We will consider two subcases.
`I-i:` $m \not\equiv 1 \pmod 3$. Note that

$$\begin{aligned} N_{2m} + N_{8m+1} &= m(14m-5) + (8m+1)(28m+1) \\ &= (14m+1)(17m+1). \end{aligned}$$

Since $\gcd(m, 14m-5) = 1$ and $\gcd(8m+1, 28m+1) = 5$, we can write, by the $a \times b$ notation,

$$\begin{aligned} f(N_{2m}) + f(N_{8m+1}) &= f\big(m \times (14m-5)\big) + f\left(\frac{8m+1}{5} \times 5(28m+1)\right) \\ &= f(m)\, f(14m-5) + f\left(\frac{8m+1}{5}\right) f(5^r). \end{aligned}$$

On the other hand, since $\gcd(14m+1, 17m+1) = \gcd(3, m-1) = 1$,

$$\begin{aligned} f(N_{2m} + N_{8m+1}) &= f\big((14m+1) \times (17m+1)\big) \\ &= f(14m+1)\, f(17m+1). \end{aligned}$$

Thus, maximal factor $17m + 1 = \frac{17}{5\cdot 28}5^r + \frac{11}{28}$ is smaller than $\beta(5^r) = \frac{1}{4} \cdot 5^r$ and $f(5^r) = 5^r$.
`I-ii:` $m \equiv 1 \pmod 3$. For this case, we use

$$\begin{aligned} N_{-2m} + N_{8m+1} &= m(14m+5) + (8m+1)(28m+1) \\ &= (14m+1)(34m+1). \end{aligned}$$

We have that $\gcd(m, 14m+5) = 1$, $\gcd(8m+1, 28m+1) = 5$, and $\gcd(14m+1, 34m+1) = \gcd(2m+3, 10) = 1$. Thus,

$$\begin{aligned} f(N_{-2m}) + f(N_{8m+1}) &= f\big(m \times (14m+5)\big) + f\left(\frac{8m+1}{5} \times 5(28m+1)\right) \\ &= f(m)\, f(14m+5) + f\left(\frac{8m+1}{5}\right) f(5^r) \end{aligned}$$

and

$$\begin{aligned} f(N_{-2m} + N_{8m+1}) &= f\big((14m+1) \times (34m+1)\big) \\ &= f(14m+1)\, f(34m+1). \end{aligned}$$

The maximal factor $34m+1 = \frac{17}{5\cdot 14}5^r - \frac{3}{14}$ is smaller than $\beta(5^r) = \frac{1}{4}\cdot 5^r$ and $f(5^r) = 5^r$.

`Case II:` $5^{2s+1} = 5(28m+9) \equiv 17 \pmod{28}$. We have $m \equiv 22 \pmod{25}$. We use

$$\begin{aligned} N_{-2m-1} + N_{4m+2} &= (2m+1)(7m+6) + (2m+1)(28m+9) \\ &= (2m+1)(7m+3). \end{aligned}$$

Note that

$$\begin{aligned} &f(N_{-2m-1}) + f(N_{4m+2}) \\ &\qquad = f\left(5^2 \times \frac{2m+1}{5} \times \frac{7m+6}{5}\right) + f\left(\frac{2m+1}{5} \times 5(28m+9)\right) \end{aligned}$$

and

$$f(N_{-2m-1} + N_{4m+2}) = f\big((2m+1) \times (7m+3)\big).$$

Thus, $f(5^r) = 5^r$, since the maximal factor $7m+3 = \frac{1}{5\cdot 4}5^r + \frac{3}{4}$ is smaller than $\beta(5^r) = \frac{1}{4}\cdot 5^r$.

`Case III:` $5^{2s+1} = 5(28m+25) \equiv 13 \pmod{28}$. Note that $m \equiv 0 \pmod{25}$ and

$$\begin{aligned} N_{6m+5} + N_{-12m-10} &= 3(6m+5)(7m+5) + 3(6m+5)(28m+25) \\ &= 15(6m+5)(7m+6). \end{aligned}$$

Then

$$\begin{aligned} &f(N_{6m+5}) + f(N_{-12m-10}) \\ &\qquad = f\left(5^2 \times \frac{6m+5}{5} \times \frac{3(7m+5)}{5}\right) + f\left(3 \times \frac{2m+1}{5} \times 5(28m+25)\right) \end{aligned}$$

and

$$f(N_{6m+5} + N_{-12m-10}) = f\big((6m+5) \times (7m+6)\big).$$

Thus, $f(5^r) = 5^r$, since the maximal factor $7m+6 = \frac{1}{5\cdot 4}\cdot 5^r - \frac{1}{4}$ is smaller than $\beta(5^r) = \frac{1}{5\cdot 4}\cdot 5^r$. □

5. Proof for $f(7^r) = 7^r$

We prove that f fixes 7^r. To do this we need to consider powers of 3 under some assumption, which was proved to be fixed by f in Section 3.

Theorem 4. *If $f(n) = n$ for all $n < 7^r$, then $f(7^r) = 7^r$.*

Proof. If $r = 2s + 1$, then $f(7^{2s+1}) = 7^{2s+1}$ by

$$N_{-m} + N_m = m \times \frac{7m+5}{2} + m \times \frac{7m-5}{2} = 7m^2,$$

where $m = 7^s$.

If r is even, note that $7^r = 12m + 1$ for some m. Then m is divisible by 4 and $m \equiv 0, 4 \pmod 5$. Note that

$$\begin{aligned} N_{-12m-1} + N_{9m+2} &= 6(7m+1)(12m+1) + 9(7m+1) \cdot \frac{9m+2}{2} \\ &= 15(7m+1) \cdot \frac{15m+2}{2} \end{aligned}$$

and each pair of linear factors is relatively prime. Thus, each factors can be split, but we cannot split 3 and $7m + 1$ yet.

If $7m + 1 = 3^k \times d$ for some $d \neq 1, 3$, then $d \geq 11$, since $\gcd(2, 7m + 1) = \gcd(5, 7m + 1) = 1$. In this case,

$$\begin{aligned} N_{-12m-1} &= 2 \times 3 \cdot \frac{7m+1}{d} \times d \times (12m+1) \\ N_{9m+2} &= 9 \cdot \frac{7m+1}{d} \times d \times \frac{9m+2}{2} \\ N_{-12m-1} + N_{9m+2} &= 5 \times 3 \cdot \frac{7m+1}{d} \times d \times \frac{15m+2}{2} \end{aligned}$$

and each factor is smaller than $7^r = 12m + 1$. Hence, $f(7^r) = 7^r$.

Now assume $7m + 1 = 3^k$. Then

$$7^r = 12 \cdot \frac{3^k - 1}{7} + 1 \text{ if and only if } 7^{r+1} = 12 \cdot 3^k - 5.$$

This equation has a trivial solution $r = k = 0$. If $r \geq 1$, then we obtain $k = 42\ell + 18$ by comparing both sides modulo 7^2.

Note that $3^{42} \equiv 1 \pmod{43}$ and $\operatorname{ord}_{43} 7 = 6$. Thus,

$$7^{r+1} \equiv 12 \cdot 3^{18} - 5 \equiv 28 \pmod{43}.$$

However, it has no solution for r. Hence, $7m + 1$ cannot be a power of 3 and we can conclude that $f(7^r) = 7^r$. □

6. Proof for $f(2^r) = 2^r$

To prove 2^r is fixed by f, we need to consider powers of 5.

Theorem 5. *If $f(n) = n$ for all $n < 2^r$, then $f(2^r) = 2^r$.*

Proof. Note that $2^r \equiv 1, 2, 4 \pmod 7$ holds. If $2^r = 7m+1$, then m is odd and, by $a \times b$ notation,

$$\begin{aligned} N_{-5m} + N_{-5m} &= 25m \cdot \frac{7m+1}{2} + 25m \cdot \frac{7m+1}{2} \\ &= 25m \times 2^{r-1} + 25m \times 2^{r-1} \\ &= 25m \times 2^r. \end{aligned}$$

Since $25m > 7m + 1 = 2^r$, we are not sure yet if f fixes $25m$.

Note that $f(25) = f(3)\, f(8) + f(1) = 25$. If $\gcd(25, m) = 1$, then we obtain $f(2^r) = 2f(2^{r-1}) = 2^r$ by

$$\begin{aligned} f(N_{-5m} + N_{-5m}) &= f(25)\, f(m)\, f(2^r) \\ &= f(25)\, f(m)\, f(2^{r-1}) + f(25)\, f(m)\, f(2^{r-1}). \end{aligned}$$

Otherwise, we may not be able to split $f(25m)$ into $f(25)$ and $f(m)$.

Assume that $m = 5^k \times d$ with $k \geq 1$ and $5 \nmid d$. Then

$$f(25m) = f(5^{k+2}d) = f(5^{k+2})\, f(d).$$

By Theorem 3 we have $f(5^{k+2}) = 5^{k+2}$ since

$$\beta(5^{k+2}) = \frac{1}{4} \cdot 5^{k+2} = \frac{25}{4d} \cdot m < 7m + 1 = 2^r.$$

Thus, we can deduce that $f(2^r) = 2^r$.

Similarly, we can show that $N = 2^r$ is fixed by f when $2^r \equiv 2, 4 \pmod 7$ by using the equalities:

$$\begin{aligned} N_{m+1} + N_{m+1} &= (m+1) \times \frac{7m+2}{2} + (m+1) \times \frac{7m+2}{2} \\ &= (m+1) \times 2^{r-1} + (m+1) \times 2^{r-1} \\ &= (m+1) \times 2^r \end{aligned}$$

and

$$\begin{aligned} N_{-3m-1} + N_{-3m-1} &= 3 \times (3m+1) \times \frac{7m+4}{2} + 3 \times (3m+1) \times \frac{7m+4}{2} \\ &= 3 \times (3m+1) \times 2^{r-1} + 3 \times (3m+1) \times 2^{r-1} \\ &= 3 \times (3m+1) \times 2^r. \end{aligned}$$

Note that the above equalities are written by the $a \times b$ notation. □

7. Proof for $f(p^r) = p^r$

We prove that $f(p^r) = p^r$ for primes $p \geq 11$.
`Case I:` $p^r = 14m + 1$ with $m \not\equiv 1 \pmod 3$ and $m \not\equiv 1 \pmod 5$. Assume that $f(n) = n$ for all $n < 14m + 1$. We use two equations according to the parity of m:

$$\begin{aligned} N_{-m} + N_{4m+1} &= m \cdot \frac{7m+5}{2} + (4m+1) \times (14m+1) \\ &= (7m+2) \times \frac{17m+1}{2} \end{aligned}$$

when m is odd and

$$\begin{aligned} N_m + N_{4m+1} &= \frac{m}{2} \cdot (7m-5) + (4m+1) \times (14m+1) \\ &= (7m+1) \cdot \frac{17m+2}{2} \end{aligned}$$

when m is even.

Since $\gcd(m, 7m \pm 5) = \gcd(m, 5)$,

$$m \cdot (7m \pm 5) = \begin{cases} 5^2 \times \frac{m}{5} \times \frac{7m \pm 5}{5} & \text{if } 5 \parallel m \text{ and } 5 \parallel (7m \pm 5) \\ \frac{m}{5} \times 5(7m \pm 5) & \text{if } 5 \parallel m \text{ and } 5^2 \mid (7m \pm 5) \\ 5m \times \frac{7m \pm 5}{5} & \text{if } 5^2 \mid m \text{ and } 5 \parallel (7m \pm 5) \end{cases}$$

and the maximal factor of $m(7m \pm 5)$ can be

$$\frac{5(7m+5)}{2} \quad \text{or} \quad 5(7m-5)$$

when they are powers of 5. So, f fixes $\frac{5(7m+5)}{2}$ and $5(7m-5)$ by Theorem 3, since

$$\begin{aligned} \beta\left(\frac{5(7m+5)}{2}\right) &= \frac{1}{4}\left(\frac{5(7m+5)}{2}\right) < 14m+1, \\ \beta(5(7m-5)) &= \frac{1}{4} \cdot 5(7m-5) < 14m+1. \end{aligned}$$

For the last term $(7m+2) \times \frac{17m+1}{2}$ of the first equation is fixed by f by the induction hypothesis. So, now, we check $f\big((7m+1) \cdot \frac{17m+2}{2}\big)$ of the second equation.

Since $\gcd(7m+1, 17m+2) = \gcd(m+1, 3)$, we can write

$$(7m+1) \cdot \frac{17m+2}{2} = \begin{cases} (7m+1) \times \frac{17m+2}{2} & \text{if } m \equiv 0, 1 \pmod 3 \\ 3^2 \times \frac{7m+1}{3} \times \frac{17m+2}{2 \cdot 3} & \text{if } 3 \parallel (7m+1) \text{ and } 3 \parallel (17m+2) \\ \frac{7m+1}{3} \times \frac{3(17m+2)}{2} & \text{if } 3 \parallel (7m+1) \text{ and } 3^2 \mid (17m+2) \\ 3(7m+1) \times \frac{17m+2}{2 \cdot 3} & \text{if } 3^2 \mid (7m+1) \text{ and } 3 \parallel (17m+2). \end{cases}$$

Thus, the maximal factor is $\frac{3(17m+2)}{2}$ when it is a power of 3. It is fixed by f by Theorem 2 since

$$\alpha\left(\frac{3(17m+2)}{2}\right) = \frac{14}{17}\cdot\frac{2}{3}\cdot\frac{3(17m+2)}{2} - \frac{12}{17} < 14m+1.$$

`Case II:` $p^r = 14m+3$ with $m \not\equiv 0 \pmod 3$ and $m \not\equiv 3 \pmod 5$. Assume that $f(n) = n$ for all $n < 14m+3$. We use

$$\begin{aligned} N_{-3m-1} + N_{6m+2} &= \frac{3}{2}(3m+1)(7m+4) + 3(3m+1)(14m+3) \\ &= \frac{15}{2}(3m+1)(7m+2). \end{aligned}$$

Since $\gcd(3m+1, 7m+4) = \gcd(5, m+2) = 1$, we can write

$$\frac{3}{2}(3m+1)(7m+4) = \begin{cases} 3\times\frac{3m+1}{2}\times(7m+4) & \text{if } m\equiv 1 \pmod 6 \\ (3m+1)\times\frac{3(7m+4)}{2} & \text{if } m\equiv 2 \pmod 6 \\ 3\times(3m+1)\times\frac{7m+4}{2} & \text{if } m\equiv 4 \pmod 6 \\ \frac{3m+1}{2}\times 3(7m+4) & \text{if } m\equiv 5 \pmod 6. \end{cases}$$

The only case which we cannot apply the induction hypothesis is $3(7m+4)$ when $7m+4$ is a power of 3. In this case, we can verify $f\big(3(7m+4)\big) = 3(7m+4)$ by Theorem 2 since

$$\alpha(3(7+4)) = \frac{14}{17}\cdot\frac{2}{3}\cdot 3(7m+4) - \frac{12}{17} < 14m+3.$$

`Case III:` $p^r = 14m+5$ with $m \not\equiv 2 \pmod 3$ and $m \not\equiv 0 \pmod 5$. Assume that $f(n) = n$ for all $n < 14m+5$. For this case we consider

$$\begin{aligned} N_m + N_{-2m} &= \frac{1}{2}m(7m-5) + m(14m+5) \\ &= \frac{5}{2}m(7m+1). \end{aligned}$$

Note that each pair of linear factors is relatively prime. Thus, the only factor we should check is $5(7m+1)$ when $7m+1$ is a power of 5. Since $\beta\big(5(7m+1)\big) = \frac{1}{4}\cdot 5(7m+1) < 14m+5$, f should fix $5(7m+1)$ by Theorem 3. Thus, $f(14m+5) = 14m+5$.

Other cases of $14m-t$ with $t = 1, 3, 5$ can be treated in the same ways as Cases I-III. So, we can conclude that $f(p^r) = p^r$.

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