THE PENULTS OF TAK: ADVENTURES IN IMPARTIAL, NORMAL-PLAY, POSITIONAL GAMES

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Abstract

For normal play, impartial games, we define *penults* as those positions in which every option results in an immediate win for the other player. We explore the number of tokens in penults of two positional games, Impartial Tic and Impartial Tak. We obtain a complete classification in the former case, and a partial classification in the other. We also explore winning strategies and further directions.

1. Penults

A child is learning chess. The teacher explains that the goal is to capture the king. After several plays, it is then explained that, really, one does not capture the king, but the game ends two moves prior in a special position called *checkmate*.

A math circle is exploring combinatorial games. The first activity is to explore subtract$\{1, 2, 3\}$, but it is called “Take-Away $1, 2, 3$.” The participants are working in pairs, and the facilitator walks around the room, asking what they have

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discovered so far. Several pairs excitedly report that they have solved the game: "You just need to get it down to 4!"

Two children are playing classical TIC-TAC-TOE. The first player has created a fork. The second player puts down their pencil and says, "You win."

In each of these scenarios, it is intuitive to the players that the game has already ended just before the second-to-last move. In these positions,

"wherever I go, you win."

We name these positions by leaving out the last two syllables of "penultimate," just as the players leave out the last two moves.

Definition 1. Let $\mathcal{G}$ be a position of an impartial game under normal play.

- If $\mathcal{G}$ has no options, it is *terminal*. ("Game over! You have won!")
- If $\mathcal{G}$ is not terminal, but it has a terminal option, it is an *ult*. ("I will win!")
- If $\mathcal{G}$ is not terminal, and all of its options are ults, it is a *penult*. ("Wherever I go, you will win!")

In many treatments of DOTS AND BOXES (see, for example, Chapter 1.7 of [1]), the analysis begins once the board is "filled up" and any further moves would create a three-sided box. These filled up positions are thus of some interest. Consider a new "abbreviated" game called D&B, which is played like DOTS AND BOXES, but simply terminates once any single box is complete. Then the "filled up" positions of DOTS AND BOXES are precisely the penults of D&B.

In the board game TAK [5], players take turns placing and moving stones on a rectangular board, with the goal of creating a road from one edge of the board to the opposite edge. Roads need not be straight lines, but only orthogonal connections count in the win condition.

In early 2023, Daniel Hodgins introduced the game IMPARTIAL TAK to Gord's Problem Incubator, an online discussion group. In this game, players take turns placing tokens on the empty spaces of a square board. The first player to create an orthogonal path from any edge to its opposite edge wins. The only penult on a 2×2 board is an empty one, and, up to symmetries of the square, there are only 2 penults on a 3×3 board, shown in Figure 1. Notice that the parities of these

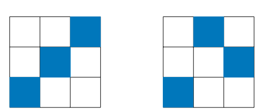


Figure 1: All 3×3 penults for IMPARTIAL TAK

penults show that the second player wins 2×2 and the first player wins 3×3.

During the course of that discussion, several games were played on a 4×4 board. To some surprise, it was discovered that penults might have 6, 7, or 8 tokens, as in Figure 2, and thus the winning strategy is much less clear. Hence the focus of this

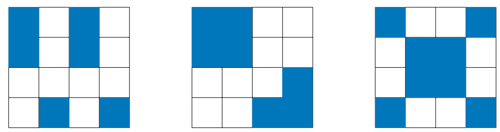


Figure 2: Some 4×4 penults for IMPARTIAL TAK with 6, 7, and 8 tokens

paper is the following.

Question. *How many tokens are in the penults of* IMPARTIAL TAK*?*

Notice that both IMPARTIAL TAK and D&B are positional games. While we defined penults more generally, these are the games for which they are the most interesting.

In Section 2 we explore winning strategies for IMPARTIAL TAK, as well as for another positional game, IMPARTIAL TIC. In Section 3, we study the penults of IMPARTIAL TIC, and in Section 4 we explore the penults of IMPARTIAL TAK.

2. Winning Strategies for Impartial Tic and Impartial Tak

Definition 2. The games IMPARTIAL TIC and IMPARTIAL TAK are both impartial, positional, and normal play. Players take turns placing tokens on empty spaces of a square board.

- In IMPARTIAL TIC, the game terminates when any row or any column is complete.
- In IMPARTIAL TAK, the game terminates when there is an orthogonal path connecting opposite edges of the board.

Let n denote the side length of the game board. For both games under consideration, the winning strategy depends on the parity of n.

Definition 3. We say a strategy S is followed *except at the end* when a player, if possible, plays to immediately win, and if not, follows S.

Proposition 1. *In* IMPARTIAL TIC*, if n is even, the second player wins via a mirroring strategy across the origin. If n is odd, the first player wins by first taking the center square, and then mirroring across the origin. In each case, the strategy should be followed except at the end.*

Proof. If n is even, the second player's moves will always result in a board position that is symmetric about the origin. Furthermore, the second player always moves in a different row and column than the preceding move. Hence, it will be the first player who first creates a row or column with $n-1$ tokens. Then the second player can take the win.

If n is odd, the proof is similar, switching the roles of the first and second players, except that we must consider the middle row and middle column separately. Each of these will have an even number of spaces left open after each move by the first player. Thus the second player cannot win by completing one of these. □

Remark 1. It is easy to check that mirroring across a center line, except at the end, will work for the even case of IMPARTIAL TIC, and will not for the odd case.

Let us now turn our attention to IMPARTIAL TAK.

Definition 4. Let a TAK-*path* be any orthogonal path from one side of an $n \times n$ board to the opposite side.

Proposition 2. *For* IMPARTIAL TAK*, if n is even, the second player wins via a mirroring strategy across a center line, except at the end.*

Proof. Suppose n is even, the second player is mirroring across the vertical center line, and that the first player has just won with a move at x on the left half of the board.

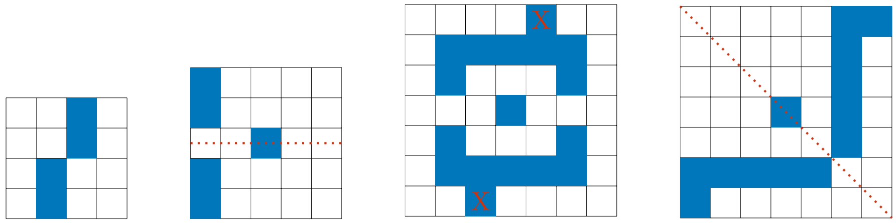
If a TAK-path connects the left and right edges, then the portion of the TAK-path on the right half, together with its mirror image, must contain a TAK-path which does not contain x, a contradiction.

Suppose there is a TAK-path that connects the top and bottom edges. Replace any portions of this path on the right side of the board with their mirror images. Then this set of tokens, S, contains a TAK-path, while $S - \{x\}$ does not. If we then also delete the second player's previous move, then either $S - \{x\}$ or its mirror image is one token short of containing a TAK-path, and the second player should have instead taken the immediate win that move. □

Mirroring across the origin will not be a winning strategy for the even case of IMPARTIAL TAK. In fact, neither of these strategies, nor mirroring across a diagonal, will work for the general odd case. See Figure 3 for counterexamples. While we suspect the first player wins for all odd n, we can only prove it for $n = 5$.

Proposition 3. *For* IMPARTIAL TAK*, if $n = 5$, the first player wins by first taking the center square, and then mirroring across the origin, except at the end.*

Proof. Suppose $n = 5$, the first player is following this strategy, and the second player has just won with the move (a, b), and that the previous move (by the first player) was at (c, d).



(a) (b) (c) (d)

Figure 3: In (a), the second player loses when mirroring across the origin on an even board. In (b), (c), (d), the first player loses on an odd board when mirroring across a center line, the origin, and a diagonal, respectively, even if they follow the strategy except at the end. In (c), the tokens marked with an X were the last two moves.

If there is a TAK-path which contains the center square, then there is also one symmetric about the origin. This one does not contain (a, b), a contradiction.

If there is a TAK-path disconnected from its mirror image, then the first player could have won on their previous move by instead moving at either (a, b) or $(n - a, n - b)$, whichever makes a connected component with more tokens.

The remaining case is that the only TAK-path is both disconnected from the middle and connected to its mirror image. On a 5×5 board, this means that if we also put a token on $(n - a, n - b)$, then all four edges of the board must be connected by a single connected component which excludes the middle, as in Figure 4(a). Notice that if n were larger, then this would not necessarily be true, as in Figure 3(c).

Thus, deleting the tokens at (a, b) and $(n - a, n - b)$ will either leave a pair of opposite edges connected, a contradiction, or will leave two connected components, each one token short of being a TAK-path. In this latter case, $\{(a, b), (n - a, n - b)\}$ must look like either the pair of orange or the pair of red tokens in Figure 4(b). Now if we remove these two tokens, as well as (c, d), the position is an ult. Hence the first player should have chosen a winning move instead of (c, d), a contradiction. □

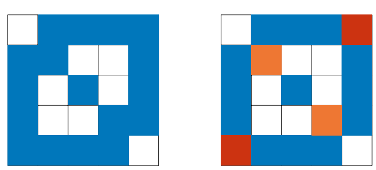

(a) (b)

Figure 4: Final case for proof of Proposition 3

Conjecture 1. If $n > 5$ is odd, then the first player has a winning strategy for IMPARTIAL TAK.

3. The Penults of (the Dual of) Impartial Tic

When exploring the penults of IMPARTIAL TIC, we will find it helpful to instead draw penults for the dual game. In this game, we begin with a board full of tokens, players take turns removing tokens from the board, and the game ends when any row or column is empty. A penult then becomes a position satisfying

- Each row and column has at least two tokens, and
- No token is in both a row and column with at least 3 tokens.

Figure 5 shows examples of several penults for $n = 5$. We use these particular constructions in the proofs below, and they are described as follows. Call the top-left-bottom-right diagonal the *main diagonal.* Then

- $\mathcal{A}_n$ is the n-by-n board which has tokens in the main diagonal, just below the main diagonal, and one in the upper right corner. Then $\mathcal{A}_n$ has $2n$ tokens and defines a penult when $n \geq 2$.
- $\mathcal{B}_n$ is the n-by-n board with tokens in the main diagonal, top row, and left column, excepting the top left corner. Then $\mathcal{B}_n$ has $3(n-1)$ tokens and defines a penult when $n \geq 3$.
- $\mathcal{C}_n$ is the n-by-n board with tokens in the left two columns and upper two rows, excepting the upper-leftmost 2×2 square. So $\mathcal{C}_n$ has $4(n-2)$ tokens and defines a penult when $n \geq 4$.

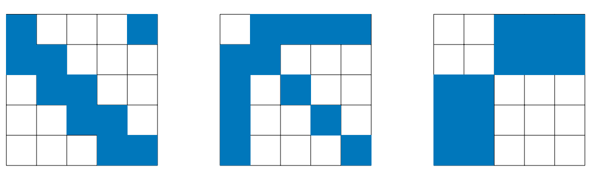


Figure 5: The penults $\mathcal{A}_5$, $\mathcal{B}_5$, and $\mathcal{C}_5$.

For $n = 2$, we obtain the unique penult, which has 4 tokens. For $n = 3$, $\mathcal{A}_3$ and $\mathcal{B}_3$ are isomorphic up to row/column permutations. In fact, this is the unique penult for $n = 3$. For $n = 4$, note that $\mathcal{A}_4$ and $\mathcal{C}_4$ have 8 tokens, while $\mathcal{B}_4$ has 9 tokens.

For $n \geq 5$, the least of these values is $2n$, and the greatest is $4(n-2)$. Since any penult must have at least 2 tokens in each row, the lower bound of $2n$ is sharp. In fact, the upper bound in these examples is sharp as well.

Proposition 4. *If $n \geq 5$, the number of tokens in a penult of the dual of* IMPARTIAL TIC *is at most $4(n-2)$. If $n = 4$, then there are at most 9 tokens.*

As a game of IMPARTIAL TIC is unchanged by any row or column permutation, it does not rely on the geometry of the board in the way IMPARTIAL TAK does, even though we use diagrams for illustration. Hence the following proof has a "purely combinatorial" flavor.

Proof of Proposition 4. Suppose we have a penult on an $n \times n$ board, and the conclusion of the proposition is not satisfied. Call a token an R-token if it is in a row with at least 3 tokens. We claim that over half of the tokens must be R-tokens. By symmetry, it will also be true that over half of the tokens must be in a column with at least 3 tokens. Then by the pigeonhole principle, there is a token that is in both a row and a column with at least 3 tokens, contradicting that the position is a penult.

To prove the claim, first note that since the position is a penult, any row containing a non-R-token must have exactly 2 tokens in it.

In the first case, suppose that $n = 4$, and assume that there are at least 10 tokens. If at least 5 are not R-tokens, then these must be in at least 3 of the rows. So we must have 3 rows with exactly 2 tokens each, and 1 row with 4 tokens. In this case, one of the columns must have at least 3 tokens, so the token in this column that is also in the filled row shows that this position is not a penult.

Now suppose that $n \geq 5$, and assume that there are at least $4n - 7$ tokens. If at least $2n - 3$ are not R-tokens, then these must be in at least $n - 1$ of the rows. This means there are $2n - 2$ tokens in these rows, leaving only one row for the remaining $2n - 4 > n$ tokens, a contradiction. □

Thus for $n = 4$, $\mathcal{A}_4$ and $\mathcal{B}_4$ show the full range of values. For $n \geq 5$, we can construct penults for each intermediate value, as well.

Proposition 5. *In the dual of* IMPARTIAL TIC*, if $n \geq 5$ and $2n \leq x \leq 4(n-2)$, we can construct a penult on an $n \times n$ board with x tokens.*

Proof. For $9 \leq m \leq 13$, we define particular penults $\mathcal{D}_{n,m}$ in which there is a fixed pattern of tokens in the top left and tokens in the remainder of two adjacent rows and two adjacent columns. See Figure 6 for examples when $n = 7$. Since a penult must have at least two tokens in each row and column, $\mathcal{D}_{n,9}$ is defined for $n \geq 5$, while $\mathcal{D}_{n,10}$, $\mathcal{D}_{n,11}$, and $\mathcal{D}_{n,12}$ are defined for $n \geq 6$, and $\mathcal{D}_{n,13}$ is defined for $n \geq 7$.

We claim that under these conditions, each $\mathcal{D}_{n,m}$ has $4n - m$ tokens. To show this, note that

$$\begin{aligned}
\mathcal{D}_{n,9} &\text{ has } 4(n-3)+3 = 4n-9 \text{ tokens,}\\
\mathcal{D}_{n,10} &\text{ has } 2(n-4)+2(n-3)+4 = 4n-10 \text{ tokens,}\\
\mathcal{D}_{n,11} &\text{ has } 4(n-4)+5 = 4n-11 \text{ tokens,}\\
\mathcal{D}_{n,12} &\text{ has } 4(n-4)+4 = 4n-12 \text{ tokens, and}\\
\mathcal{D}_{n,13} &\text{ has } 4(n-5)+7 = 4n-13 \text{ tokens.}
\end{aligned}$$

For $n = 5$, the proposition is proved by considering $\mathcal{A}_5$, $\mathcal{D}_{5,9}$, and $\mathcal{C}_5$. For $n = 6$, consider $\mathcal{A}_6$, $\mathcal{D}_{6,11}$, $\mathcal{D}_{6,10}$, $\mathcal{D}_{6,9}$, and $\mathcal{C}_6$. Now let $n \geq 7$.

For $3 \leq k \leq n-3$, the otherwise empty $n \times n$ board with $\mathcal{A}_k$ in the upper left corner and $\mathcal{B}_{n-k}$ in the lower right corner is a penult. This penult has $2k + 3(n-k-1) = 3n-k-3$ tokens. These penults thus have all possible numbers of tokens in the interval $[2n, 3n-6]$.

For $3 \leq k \leq n-4$, the otherwise empty $n \times n$ board with $\mathcal{B}_k$ in the upper left corner and $\mathcal{C}_{n-k}$ in the lower right corner is a penult. This penult has $3(k-1) + 4(n-k-2) = 4n-k-11$ tokens. These penults thus have all possible numbers of tokens in the interval $[3n-7, 4n-14]$.

As the various $\mathcal{D}_{n,m}$ have $4n-13, \ldots, 4n-9$ tokens, respectively, and $\mathcal{C}_n$ has $4n-8 = 4(n-2)$ tokens, the proposition is proved for $n \geq 7$ as well. □

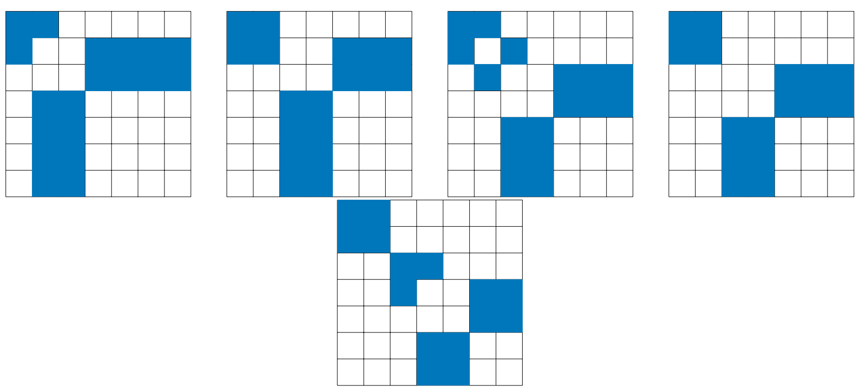


Figure 6: The penults $\mathcal{D}_{7,9}$, $\mathcal{D}_{7,10}$, $\mathcal{D}_{7,11}$, $\mathcal{D}_{7,12}$, $\mathcal{D}_{7,13}$

Corollary. *The possible number of tokens in a penult on an $n \times n$ board of the dual game of* IMPARTIAL TIC *is given in Table 1.*

n	set of possible number of tokens in a penult
2	$\{4\}$
3	$\{6\}$
4	$\{8, 9\}$
≥ 5	$[2n, 4(n-2)]$

Table 1: Summary of penults of the dual game of IMPARITAL TIC

4. The Penults of Impartial Tak

4.1. Initial Constructions.

We now turn our attention to the penults of IMPARTIAL TAK. For $n \geq 1$, let $L(n)$ and $U(n)$ denote the number of tokens in the smallest and largest penult on an $n \times n$ board. Clearly, each row of a penult must be missing at least two tokens. Hence $U(n) \leq n^2 - 2n$. On the other hand, we will soon see that this bound is sharp.

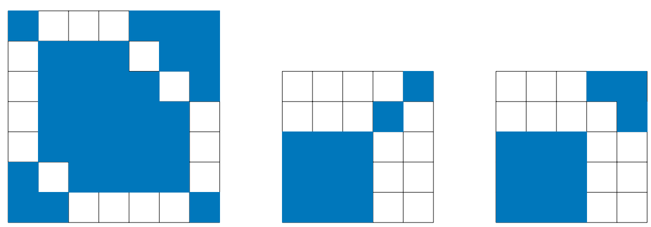


Figure 7: The Variable Diamond with n = 7, (k, l) = (4, 5), and the L-Snakes for n = 5

Proposition 6. *Let $n \geq 4$ and $n^2 - 4(n-2) - 2 \leq x \leq n^2 - 2n$. Then there is a penult on an $n \times n$ board with x tokens.*

Proof. Consider the penults shown in Figure 7. The L-Snakes are constructed by placing 2 or 3 tokens in the top-right corner, and all tokens in the lower-left $(n-2) \times (n-2)$ square. Hence they have $(n-2)^2 + 2 = n^2 - 4(n-2) - 2$ and $(n-2)^2 + 3 = n^2 - 4(n-2) - 1$ tokens, respectively.

Now consider the Variable Diamond, with parameters (k, l) for some $3 \leq k, l \leq n-1$. This is constructed by starting with a complete $n \times n$ grid, and eliminating tokens in the following positions:

- from the top row, positions 2 through k;
- from the right column, positions $n - k + 1$ through $n - 1$;
- the $(n - k - 1)$ positions diagonally connecting these two regions; *and*

- from the left column, positions 2 through l;
- from the bottom row, positions $n-l+1$ through $n-1$;
- the $(n-l-1)$ positions diagonally connecting these two regions.

Then the number of tokens is

$$n^2 - 2(k-1) - (n-k-1) - 2(l-1) - (n-l-1) = n^2 - 2n - k - l + 6,$$

where $3 \leq k, l \leq n-1$. So the possible numbers of tokens contains the interval $[n^2 - 4(n-2), n^2 - 2n]$. □

Corollary 1. *For all $n \geq 2$, we have $U(n) = n^2 - 2n$.*

We now turn our attention to the lower bound.

Proposition 7. *Suppose $n \geq 6$. There is a penult on an $n \times n$ board with*

$$\begin{cases} 2n + (n+2)(\frac{n-4}{3}) & = \frac{n^2}{3} + \frac{4}{3}n - \frac{8}{3} \text{ if } n \equiv 1 \pmod 6 \\ 2(n-2) + n(\frac{n-2}{3}) & = \frac{n^2}{3} + \frac{4}{3}n - 4 \text{ if } n \equiv 2 \pmod 6 \\ (n-2) + (n-1) + (n+1)(\frac{n}{3} - 1) & = \frac{n^2}{3} + \frac{4}{3}n - 4 \text{ if } n \equiv 3 \pmod 6 \\ 2n + (n+2)(\frac{n-4}{3}) & = \frac{n^2}{3} + \frac{4}{3}n - \frac{8}{3} \text{ if } n \equiv 4 \pmod 6 \\ 2n + (n-2) + (n+2)(\frac{n-5}{3}) & = \frac{n^2}{3} + 2n - \frac{16}{3} \text{ if } n \equiv 5 \pmod 6 \\ 3(n-2) + n(\frac{n-3}{3}) & = \frac{n^2}{3} + 2n - 6 \text{ if } n \equiv 0 \pmod 6 \end{cases}$$

tokens. Hence, $L(n) \leq \frac{n^2}{3} + 2n - \frac{8}{3}$ for all $n \geq 6$.

Proof. See the Snake Diagrams in Figure 8. In the case of $n \equiv 3 \pmod 6$, we will explain how to draw the corresponding Snake Diagram (shown for $n = 15$), and how to enumerate the tokens. The other cases are similar.

Suppose $n \equiv 3 \pmod 6$. To construct the corresponding Snake Diagram, place tokens in the following squares:

- in the bottom row, the first $n-2$ squares;
- in the first column, the squares in rows 3 through $n-1$, and also in squares $(2,2)$ and $(1,3)$;
- in columns numbered $m \equiv 1 \pmod 6), (m > 1)$, squares in rows 3 through $n-1$, and also in squares $(2, m+1)$, $(1, m+2)$, $(2, m-1)$, $(1, m-2)$;
- in columns numbered $m \equiv 4 \pmod 6$, squares in rows 1 through $n-3$, and also in squares $(n-2, m+1)$, $(n-1, m+2)$, $(n-2, m-1)$, $(n-1, m-2)$.

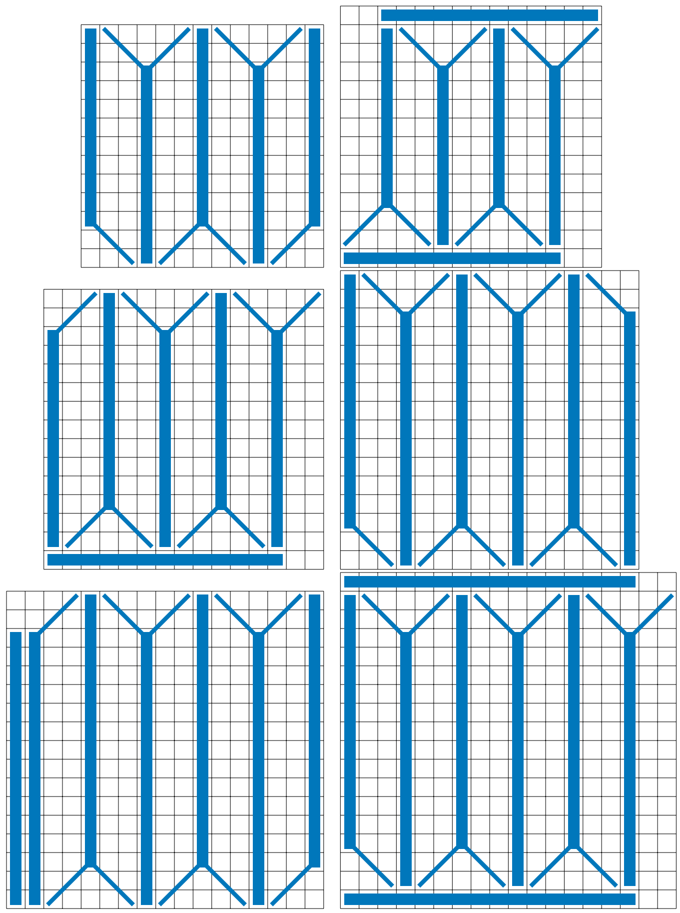
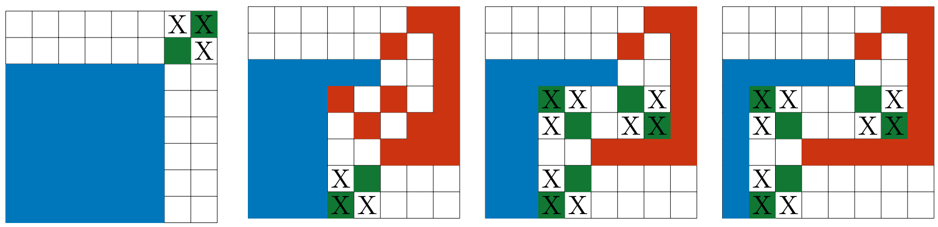


Figure 8: The Snake Diagrams for $n = 13, 14, 15, 16, 17, 18$, drawn to emphasize the enumeration of the tokens.

The first item adds $n - 2$ tokens. The second item adds $n - 1$ tokens. The last two items add $n + 1$ tokens for each $m \equiv 1 \pmod 3$, $1 < m < n$, for a total of $(n + 1)(\frac{n}{3} - 1)$ tokens. $\square$

We are pretty sure the L-Snakes can be iteratively transformed into the Snake Diagrams via a sequence of penults. Figure 9 shows one way to do it for $n = 8$. The first picture shown is the two L-Snakes overlaid on each other, one with the green squares, and one with the Xs instead. This gives 38 or 39 tokens. Each of the succeeding pictures similarly shows several variations, where the lowest number of tokens indicated is obtained by including all green squares, and the highest number obtained by instead including all the Xs. However, we do not see a non-tedious way to verify this in general, so we mark the following as a conjecture.



(a) (b) (c) (d)

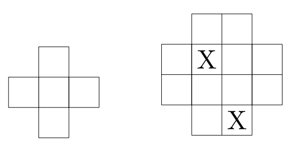
Figure 9: Number of tokens (a) 38/39, (b) 36/37, (c) 32/33/34/35, (d) 28/29/30/31

Conjecture 2. For all $n \geq 6$, let M_n be the value given by Proposition 7, and let $M_n \leq x \leq n^2 - 4(n-2) - 2$. Then there is a penult on an $n \times n$ board with x tokens.

We also think that the snake diagrams are essentially minimal.

Conjecture 3. For all n, we have $L(n) \geq \frac{n^2}{3} + an + b$, for some constants a and b.

4.2. From Snakes to Crosses: Increasing the Lower Bound



(a) (b)

0	1	1	1	0
1	1	2	1	1
1	2	2	2	1
1	1	2	1	1
0	1	1	1	0

(c)

Figure 10: (a) The Cross, (b) the Thick Cross, and (c) the Weighted Cross

We next show how to use the gadgets shown in Figure 10 to obtain some partial progress toward Conjecture 3.

Proposition 8. *Consider any position on an $n \times n$ IMPARTIAL TAK board, and any sub-board C in the shape of the Cross. If the position is a penult, then C must contain at least one token.*

Proof. Suppose indeed that C contains no tokens and that the position is a penult. Suppose the players are Alice and Bob, and that it is Alice's turn. Let Alice move in the middle square of C. Since the position was a penult, Bob can now win.

If Bob wins by placing a token outside C, then his move creates a TAK-path using only squares outside C. Hence Alice could have won by moving in Bob's square instead, contradicting that the original position was a penult.

On the other hand, suppose Bob can now win by placing a token within C. This means that, after his move, there is a TAK-path which uses Bob's latest move. However, it is clear that this TAK-path does not require Alice's move! Hence, Alice could have won by moving in Bob's square instead, contradicting that the original position was a penult. □

Now any $n \times n$ board can be packed with Crosses, covering all but the first and last couple of rows and columns. Hence we have proved the following.

Proposition 9. *Any penult must have at least $\frac{1}{5}n^2 + O(n)$ tokens.*

The $O(n)$ term can be made more specific by analyzing how to efficiently pack an $n \times n$ board with Crosses. We will instead improve upon this below by increasing the leading constant.

Before we do, let us try to understand in what way Alice had a "free" move in the above discussion, contradicting the maximality of the posited penult position with an empty Cross. Notice that any minimal TAK-path which uses tokens in C must, within C, be an orthogonal path from one boundary of C to another (not necessarily opposite) boundary of C. So Alice's "free" move is not allowed to immediately create a new connection between boundaries of C, otherwise she might have a winning move. Thus the only candidate is the center of C. So what of Bob? Bob *can* create a new orthogonal path from one boundary of C to another: simply move in a non-center square of C. But if this creates a new TAK-path and causes Bob to win, then Alice could have stolen Bob's move. This is because it was not a new *opportunity* created by Alice's move! Hence the key property of Alice's "free" move in the center of C is the following.

Definition 5. In a particular position of IMPARTIAL TAK, in a sub-board C, a move for Alice is a *free move* if: Alice's move does not immediately create a new orthogonal path between boundaries of C, and it does not create, for Bob, a new opportunity to create a new orthogonal path from one boundary of C to another.

Proposition 10. *If Alice has a free move, then the position is not a penult.*

Using this property, we can move on to the Thick Cross.

Proposition 11. *Consider any position on an $n \times n$ IMPARTIAL TAK board, and any sub-board C in the shape of the Thick Cross. If the position is a penult, then C must contain at least 3 tokens.*

Proof. Suppose a penult contains a Thick Cross with only two tokens. If both tokens are in the outer squares, then there will be a sub-Cross with no tokens, contradicting Proposition 8. If both tokens are in the middle squares, then Alice has a free move in one of the other middle squares. If one token is in a middle square and one in a boundary square, then to avoid an empty Cross, they must be placed as in Figure 10. In this case, Alice again has a free move in the middle. □

Packing an $n \times n$ board with Thick Crosses, each of which has 12 squares, we now have the following.

Proposition 12. *Any penult must have at least* $\frac{3}{12}n^2 + O(n) = \frac{1}{4}n^2 + O(n)$ *tokens.*

The next two propositions show how to use the Weighted Cross to obtain an even better lower bound.

Proposition 13. *Consider any position on an* $n \times n$ IMPARTIAL TAK *board, and any* 5×5 *sub-board* C*. If the position is a penult, then the tokens in* C *must be arranged so that the corresponding numbers in Figure 10(c) add up to at least* 7*.*

Proof. Suppose a penult contains a Weighted Cross whose tokens value at most 6. There are at most 3 tokens of value 2, and we analyze by cases, starting with the extremes.

`Case A:` there are 0 tokens of value 2. Then there is an empty cross in the middle, contradicting Proposition 8.

`Case B:` there are 3 tokens of value 2. Then there are none of value 1. Hence either of the other squares of value 2 are a free move for Alice. See Figure 11, for example.

0				0
		*		
	2	2	*	
		2		
0				0

Figure 11: Either of the other squares of value 2 are a free move for Alice.

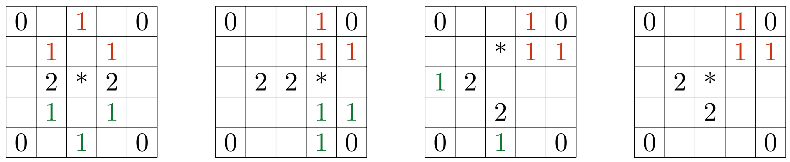
In the remaining cases and subcases we combine these two techniques. In each diagram shown, in addition to the tokens on the black numbers, there must be at least one token on each color shown, and the $*$ will indicate where Alice has a free move.

`Case C:` There are two tokens of value 2. Then there are at most two of value 1, maybe some of value 0, and we analyze the subcases by the relative positions of the tokens of value 2, as in Figure 12.

`Case Ca:` the two tokens of value 2 are as shown. In this case, to avoid empty sub-Crosses, there must be tokens in at least one of the red and at least one of the green 1s. Then Alice has a free move at $*$.

`Case Cb:` as shown. The proof is similar to the previous case.

`Case Cc:` the two tokens of value 2 are as shown. In this case, to avoid an empty sub-Cross, there must be tokens in at least one of the red positions. If there is a token on one of the green positions, Alice has a free move as shown in Case Cc1. Otherwise, Alice has a free move as shown in Case Cc2.



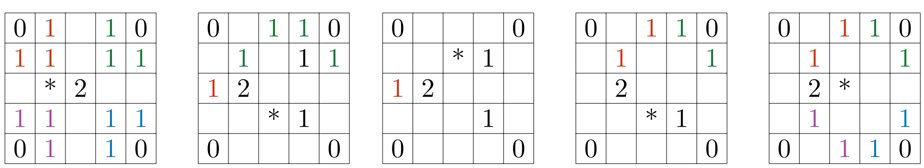
(i) (ii) (iii) (iv)

Figure 12: (i) Case Ca, (ii) Case Cb, (iii) Case Cc1, and (iv) Case Cc2

`Case D:` there is one token of value 2. Then there are at most four of value 1, maybe some of value 0, and we analyze the subcases as in Figure 13. First we describe the subcases. Case Da: The token of value 2 is in the center. Case Db: The token of value 2 is not in the center. Case Db1: There are also tokens on both of the black 1s shown. Case Db2: There is also a token on *one* of the black 1s. Case Db3: There are not tokens on either of the black 1s.

In Cases Da, Db2, Db3, to avoid empty sub-Crosses, there must be a token on at least one of the red (and green/blue/yellow) 1s. Then Alice has a free move at $*$.

Case Db1 is slightly different. In this case, we can deduce that there must be a token on the red 1 shown, else Alice will have a free move in the center. Now there is at most one more possible token of value 1. If it is one of the green ones shown in Case Db1a, then Alice has a free move as shown there. Otherwise, Alice has a free move as shown in Case Db1b. □



(i) (ii) (iii) (iv) (v)

Figure 13: (i) Case Da, (ii) Case Db1a, (iii) Case Db1b, (iv) Case Db2, and (v) Case Db3

Proposition 14. *If $n \geq 5$, a penult on an $n \times n$ board must have at least $\frac{7}{26}(n-4)^2$ tokens.*

Proof. Our strategy is to use a probabilistic argument. First, notice that the sum of the numbers shown in the Weighted Cross is 26.

Next consider any penult on an $n \times n$ board with T tokens. For each $1 \leq i, j \leq n$, let $x_{i,j} = 1$ if there is a token in position (i, j) and 0 if not. Then $T = \sum_{1 \leq i,j \leq n} x_{i,j}$.

On the other hand, for $1 \leq r, c \leq (n-4)$, consider the corresponding sub-Weighted Cross whose lower left corner is at (r, c). By Proposition 13, the corresponding weighted sum is at least 7:

$$\begin{array}{lllllllll} 0x_{r,c} & + & x_{r,c+1} & + & x_{r,c+2} & + & x_{r,c+3} & + & 0x_{r,c+4} \\ +\ x_{r+1,c} & + & x_{r+1,c+1} & + & 2x_{r+2,c+2} & + & x_{r+1,c+3} & + & x_{r+1,c+4} \\ +\ x_{r+2,c} & + & 2x_{r+2,c+1} & + & 2x_{r+2,c+2} & + & 2x_{r+2,c+3} & + & x_{r+2,c+4} \\ +\ x_{r+3,c} & + & x_{r+3,c+1} & + & 2x_{r+3,c+2} & + & x_{r+3,c+3} & + & x_{r+3,c+4} \\ +\ 0x_{r+4,c} & + & x_{r+4,c+1} & + & x_{r+4,c+2} & + & x_{r+4,c+3} & + & 0x_{r+4,c+4} \end{array}$$
$$\geq 7.$$

Now if we sum over $1 \leq r, c \leq (n-4)$, the right-hand side becomes $7(n-4)^2$. For the left-hand side, for each $1 \leq i, j \leq n$, there will be a term $a_{i,j}x_{i,j}$, where $a_{i,j} = 26$ if $5 \leq i, j \leq (n-4)$, and $a_{i,j} < 26$ otherwise. Hence we have

$$26T = \sum_{1 \leq i,j \leq n} 26x_{i,j} \geq \sum_{1 \leq i,j \leq n} a_{i,j}x_{i,j} \geq 7(n-4)^2.$$

In other words, the total number of tokens in the penult is at least $\frac{7}{26}(n-4)^2$. □

So we have improved the leading constant in Proposition 8 from 0.2 to $\frac{7}{26} \approx 0.269$, closer to Conjecture 3's value of $\frac{1}{3}$.

4.3. Computation

We have computed all penults up to 6×6 using Maple. As expected, $L(4) = 6$. We then saw that $L(5) = 10$ and $L(6) = 16$, and that the possible numbers of tokens in each case are intervals. There are a total of 59 nonisometric 4×4 penults, and 3629 nonisometric 5×5 penults. Curiously, there is a *unique* 5×5 penult with 10 tokens (see Figure 14), and this is the only size board and number of tokens for which we have observed this uniqueness! See the last author's website, [6], for all relevant Maple code.

5. Further Adventures

5.1. Penult Intervals and the Game D&B

Is there a natural example of a impartial, normal-play, positional game in which the possible numbers of tokens in a penult on a fixed board size does *not* form an

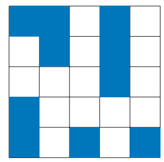


Figure 14: The unique minimal 5×5 penult

interval? At first glance, D&B also seems to follow the same pattern as IMPARTIAL TIC and IMPARTIAL TAK. Representative penults for the smallest cases are shown in Figure 15. Note that "tokens" in this game are line segments.

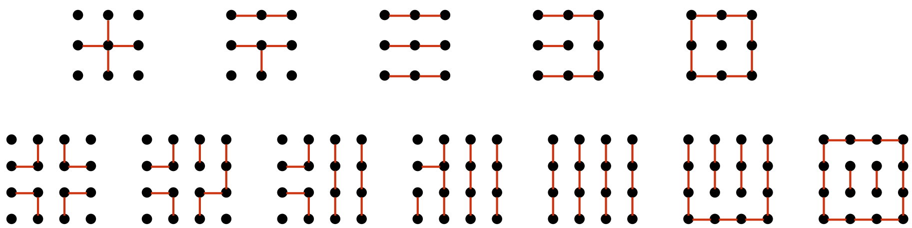


Figure 15: A penult for D&B on a 3×3 board can have 4, 5, 6, 7, or 8 tokens. A penult for D&B on a 4×4 board can have from 8 to 14 tokens.

5.2. Other Remoteness Numbers

In the larger theory of combinatorial games, the *remoteness* of a game position (due to Smith [4]) is the greatest number of moves the current player can force if the position is a losing one, or the least number of moves they can force if the position is a winning one. In other words, terminal positions have remoteness 0, ults have remoteness 1, and penults are precisely the game positions with remoteness 2. Is our analysis of penults also interesting for positions of some other fixed remoteness?

5.3. Partizan Games and Non-Positional Games

Definition 1 is not restricted to positional games. It can also be extended to partizan games as follows.

Definition 6. Let $G = \{L \mid R\}$ be a game position.

- G is a *left-penult* if $R \neq \emptyset$, and for all $\{A \mid B\} \in R$, $\emptyset \in A$.
- G is an *right-penult* if $L \neq \emptyset$, and for all $\{A \mid B\} \in L$, $\emptyset \in B$.
- G is a *penult* if it is both a left-penult and a right-penult.

The left and right penults of Quarto [3], for example, might be worth a look.

5.4. Misère Play

Is there a reasonable notion of a penult for misère-play games? For example, Notakto [2] is a positional game canonically played in misère.

Acknowledgement. We are grateful to the anonymous reviewer for many helpful comments in an earlier draft.

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