VARIATIONS OF THE NIM GAME PLAYED ON A CUBE

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Abstract

The Sprague–Grundy theory makes any short impartial combinatorial game equivalent to the game Nim, because of which Nim has many generalizations and modifications. This paper introduces new versions of Nim, where two players play Nim on a cube. Tokens are allocated to each face of the cube, and we define different game variations based on the moves available to the players. These variations include selecting a face, an edge, or a vertex of the cube. We consider one more variation by selecting one of the cube's rotations. We provide the winning set characteristics for the second player in all these variations under both normal and misère play conventions. Additionally, we define the slow versions of these games and analyze them under normal play.

1. Introduction

The Nim game has existed for several centuries, possibly originating in China, but became popular in Europe during the 16th century. In Nim, there are multiple stacks of tokens, known as heaps, and two players alternately take one or more tokens from a single heap. The game ends when all the heaps are empty. The player who makes the last move wins, which is called the *normal play* convention. Another playing convention is the one in which the player forced to make the last move loses, called the *misère play*. Its current name, "Nim" is given by Charles L. Bouton, who developed a complete mathematical theory for both normal play convention and misère play convention in 1901 [3]. Since then, many variants of this game have been proposed and analyzed in the literature [2], and the well studied variants are Wythoff's games [10], subtraction games [1], and Moore's k-Nim [7].

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Another variation is Nim played on graphs [2], where heaps are placed on the edges. On his turn, the player selects a vertex of the graph and removes one or more tokens from the union of the edges, with one end of the edge being the selected vertex. *Circular Nim* [4] is a particular variation of this graph Nim, where we arrange heaps on the edges of a cycle graph with n vertices, and a move is to select the k-consecutive edges and remove at least one token in total from them, and the game is denoted by $CN(n, k)$. *Slow Nim* [6] is another variation of Nim where one restricts the number of tokens removed from the chosen heaps. In Slow Nim, the player selects the heaps according to the rule of the game and is allowed to remove at most one token from selected heaps and must reduce tokens in at least one selected heap by one.

In this paper, we introduce Nim games on a cube. We assign tokens to the faces of the cube, and then we define different variants by prescribing various ways to select the faces of the cube. The rules of these variants are as follows:

1. Assign a non-negative integer (number of tokens) to each face of the cube, as in Figure 1, with the notation for the face with a tokens as f_a.

2. Two players proceed in the game by alternate moves.

3. We define four different game variants according to the moves available to the players:

   - CV (Cube Vertex): The player selects one of the cube's vertices and removes at least one token in total from the three faces intersecting at the selected vertex.
   - CE (Cube Edge): The player selects one of the edges of the cube and removes at least one token in total from the two faces that share the selected edge.
   - CF (Cube Face): The player selects one face of the cube and removes at least one token from that face.
   - CR (Cube Rotation): The cube has three rotations: $f_a - f_b - f_A - f_B$, $f_a - f_c - f_A - f_C$, and $f_b - f_c - f_B - f_C$. On his turn, the player selects one of these rotations and removes tokens from the faces in the selected rotation.

4. The game ends when no legal move is available.

5. The winner is decided according to the normal or misère play convention.

This paper is organized as follows. Section 2 discusses the required preliminaries to analyze Nim games on the cube. In Section 3, we provide the winning set characteristics for the second player in the above four variations under normal play.

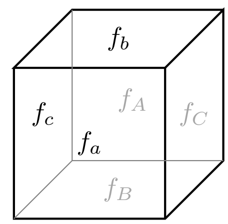

Figure 1: Cube with tokens on faces

We analyze the misère play versions of all four variants in Section 4. In Section 5, we define slow versions of the Nim games on the cube, giving the winning set characteristics of the second player under normal play. In Section 6, we propose another variant, where tokens are assigned to edges or vertices of the cube, and we also provide partial analysis along with some open problems.

2. Preliminaries

A *combinatorial game* is a game with complete information (perfect information), and there is no chance involved in the gameplay. Each player is aware of the game's position at any stage of the game. Two-player combinatorial games can be classified as impartial and partizan games.

Definition 1. A game in which two players take turns making moves is:
(i) *impartial* (or non-partizan) if, from each position in the game, the same options are available to both players;
(ii) *short* if, positions in the game are never repeated, only a finite number of other positions can be reached, and the game terminates after a finite number of moves.

Combinatorial games are studied in terms of determining the winner when playing from a given initial position. A short impartial game is a two-player game that terminates after a finite number of moves, in which both players have identical options from every position. Since the game is finite, deterministic, and involves no chance or hidden information, it admits neither draws nor cycles. Consequently, from each position, exactly one of the two players has a winning strategy under optimal play. We make the following two definitions: an $\mathcal{N}$*-position* is an initial position from which the first player has a winning strategy, meaning the next player to move from this position can force a win. In contrast, a $\mathcal{P}$*-position* is an initial position from which the second player wins, meaning the player who previously moved to this position can force a win.

In the 1930s, Sprague [9] and Grundy [5] created a comprehensive theoretical framework to analyze all short impartial games. The Sprague–Grundy theory gives us a characterization to identify whether the given position is in $\mathcal{P}$ or in $\mathcal{N}$. Instead of classifying every position in the game one by one into two classes, combinatorial game theory aims to identify the patterns that allow us to quickly determine whether any given position of the game is in $\mathcal{P}$ or in $\mathcal{N}$. The following partition theorem for short impartial games is useful to verify that the identified pattern for class $\mathcal{P}$ is correct.

Theorem 1 (Partition Theorem for Impartial Games [1])**.** *Suppose that all possible initial positions of the given short impartial game can be partitioned into two mutually disjoint sets, W (winning) and L (losing), with the following properties:*

- *The terminal positions are in W,*
- *Every legal option from the position in W leads to L,*
- *Every position in L, has at least one legal option which leads to W.*

Then W is the set of $\mathcal{P}$-positions, and L is the set of $\mathcal{N}$-positions.

In the next section, we analyze Nim games on the cube under normal play using the above theorem.

3. Nim Games on a Cube under Normal Play

Consider a cube with the assigned number of tokens a, b, c, A, B, and C on the faces f_a, f_b, f_c, f_A, f_B, and f_C, respectively, with face pairs (f_a, f_A), (f_b, f_B), and (f_c, f_C) opposite to each other as shown in Figure 1. The common edge between faces f_a, f_b is denoted by ab as in Figure 2, and the cube's vertex formed by intersecting three faces f_a, f_b, f_c is denoted by abc as shown in Figure 2. Due to the symmetry in the cube's structure, when we select a vertex, an edge, a face, or a rotation, we can assume that the number of tokens on faces follows $a \leq A,\ b \leq B$, and $c \leq C$.

First, we consider the game variant CV, where, on his turn, the player will select a vertex and remove tokens from the faces incident at the selected vertex.

Theorem 2. *In the game CV, the set of second-player winning positions is*

$$\mathcal{P} = \{(a, b, c, A, B, C) \mid a = A,\ b = B,\ c = C\}.$$

Proof. Let $W = \{(a, b, c, A, B, C) \mid a = A,\ b = B,\ c = C\}$. The terminal position of this game is $(0, 0, 0, 0, 0, 0) \in W$.

Suppose that $p = (a, b, c, a, b, c) \in W$. Observe that, for a position p, each cube vertex is an intersecting point of three faces with tokens a, b, and c. Then, every legal move from the position $p = (a, b, c, a, b, c)$ leads to an option of the form

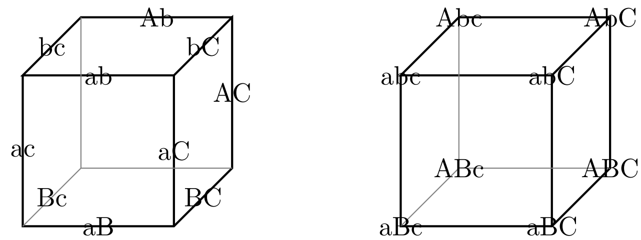

Figure 2: Edges and vertices of the cube

$p' = (a-i, b-j, c-k, a, b, c)$ with $i+j+k \geq 1$. Hence, at least one of the equations $a - i = a$, $b - j = b$, and $c - k = c$ is not satisfied, which implies that $p' \notin W$.

Start with an initial position $p = (a, b, c, A, B, C) \notin W$. Then, at least one of the inequalities $a \leq A$, $b \leq B$, and $c \leq C$ is strict. Consider the legal move by choosing the vertex ABC and removing at least one token from the faces with tokens A, B, and C and making them a, b, and c, respectively. The game comes to the position $p' = (a, b, c, a, b, c) \in W$. By Theorem 1, we conclude that $W = \mathcal{P}$. □

Next, consider the game CE, where, on his turn, the player selects an edge and removes tokens from the two faces with the selected edge as a common edge.

Theorem 3. *In the game CE, the set of second-player winning positions is*

$$\mathcal{P} = \{(a, b, c, A, B, C) \mid a \bigoplus A = b \bigoplus B = c \bigoplus C\},$$

where $\bigoplus$ denotes the Nim-sum.

Proof. Let $W = \{(a, b, c, A, B, C) \mid a \bigoplus A = b \bigoplus B = c \bigoplus C\}$. The game CE has exactly one terminal position, namely $(0, 0, 0, 0, 0, 0)$. Clearly, this position belongs to W.

Assume that $p = (a, b, c, A, B, C) \in W$. Any legal move from a position p involves selecting one of the 12 edges, say edge aB, and transitioning to a new position $p' = (a - i, b, c, A, B - j, C)$, where $i + j \geq 1$ since at least one token must be removed. This implies that either $i \geq 1$ or $j \geq 1$, which in turn guarantees that either $(a - i) \bigoplus A \neq a \bigoplus A$ or $b \bigoplus B \neq b \bigoplus (B - j)$, respectively. Hence, the resulting position $p' = (a - i, b, c, A, B - j, C)$ does not satisfy at least one of the equations $(a - i) \bigoplus A = c \bigoplus C$ or $c \bigoplus C = b \bigoplus (B - j)$, which implies that $p' \notin W$.

To establish the third condition of Theorem 1, consider a position $p \notin W$. Then, at least one of the following conditions must hold:

$$a \bigoplus A \neq b \bigoplus B, \quad a \bigoplus A \neq c \bigoplus C, \quad \text{or} \quad c \bigoplus C \neq b \bigoplus B.$$

Let $x = min(a \bigoplus A, b \bigoplus B, c \bigoplus C)$. Without loss of generality, we may assume that $x = a \bigoplus A$. Then, by the properties of Nim-sum, since $x \leq b \bigoplus B$, there exists $b' < b$ or $B' < B$, such that $x = b' \bigoplus B$ or $x = b \bigoplus B'$, respectively. Similarly, as $x \leq c \bigoplus C$, there exists $c' < c$ or $C' \leq C$, such that $x = c' \bigoplus C$ or $x = c \bigoplus C'$, respectively. Consequently, by selecting one of the edges bc, bC, Bc, and BC and making a move to $b'c'$, $b'C'$, $B'c'$, or $B'C'$, the game comes to a position where three Nim-sums of opposite faces are the same, from which it follows that $p' \in W$. $\square$

Obviously, the next variation CF is equivalent to the original Nim game with six heaps, and hence, by the following theorem of Bouton [3], we obtain the set of $\mathcal{P}$-positions for the game CF.

Theorem 4. *In the game CF, the set of second-player winning positions is*

$$\mathcal{P}= \{(a, b, c, A, B, C) \mid a \bigoplus b \bigoplus c \bigoplus A \bigoplus B \bigoplus C = 0\}.$$

Now, we analyze the variation CR, where on his move, the player selects one of the cube's rotations $f_a - f_b - f_A - f_B$, $f_a - f_c - f_A - f_C$, and $f_b - f_c - f_B - f_C$ and removes at least one token in total from the four faces in the selected rotation.

Theorem 5. *In the game CR, the set of second-player winning positions is*

$$\mathcal{P}=\{(a, b, c, A, B, C) \mid a + A = b + B = c + C\}.$$

Proof. Let $W = \{(a, b, c, A, B, C) \mid a + A = b + B = c + C\}$. The only terminal position $(0, 0, 0, 0, 0, 0)$ for this game belongs to W.

For any position $p = (a, b, c, A, B, C) \in W$, we have $a + A = b + B = c + C$. The possible moves are by selecting one of the rotations $f_a - f_b - f_A - f_B$, $f_a - f_c - f_A - f_C$, and $f_b - f_c - f_B - f_C$. By selecting the rotation $f_a - f_b - f_A - f_B$ and making any legal move on $p = (a, b, c, A, B, C)$, we get $p' = (a', b', c, A', B', C)$ with either $a' + A' \neq c + C$ or $b' + B' \neq c + C$. As a result, $p' \notin W$. Similarly, by any legal move after selecting a rotation $f_a - f_c - f_A - f_C$ or $f_b - f_c - f_B - f_C$, we get the fact that the resulting position p' does not belong to W.

Now, consider a position $p = (a, b, c, A, B, C) \notin W$. Without loss of generality, assume that $x = min(a + A, b + B, c + C) = a + A$. Then by selecting the rotation $f_b - f_c - f_B - f_C$ and making a move $p = (a, b, c, A, B, C) \rightarrow p' = (a, b' = b - i, c' = c - j, A, B' = a + A - b + i, C' = a + A - c + j)$, we get the position with $a + A = b' + B'$ and $a + A = c' + C'$, which shows that $p' \in W$. $\square$

In the next section, we study the misère version for games CV, CE, CR, and CF.

4. Nim Games on a Cube under Misère Play

To analyze Nim played on a cube under the misère play convention, we consider a position $p = (a, b, c, A, B, C)$, where, without loss of generality, we assume that $C = max(p)$, $a \leq A$, $b \leq B$, and $a \leq b$.

A winning strategy for the winning player under the misère play for the game CV is as follows. The winning player should play the game as he would play under normal play, as long as there exists at least one pair of opposite faces where each face has more than one token. When the winning player is met with a position in which any pair of opposite faces has a face with at most one token, then clearly the underlying position $p = (a, b, c, A, B, C)$ satisfies $0 \leq a, b, c \leq 1$. The player can now make a move such that the resulting position belongs to the set $\{(0, 0, 0, 0, 0, 1), (0, 1, 1, 0, 1, 1), (1, 1, 1, 1, 1, 1)\}$. The existence of such a move follows from the proof of the next theorem.

Theorem 6. *In the game CV under misère play, the set of second-player winning positions is*

$$\mathcal{P} = \{(a, b, c, A, B, C) \mid a = A,\ b = B,\ c = C \geq 2\} \\ \bigsqcup \{(0, 0, 0, 0, 0, 1), (0, 1, 1, 0, 1, 1), (1, 1, 1, 1, 1, 1)\}.$$

Proof. For a given position $p = (a, b, c, A, B, C)$, without loss of generality, we assume that, $C = max(p)$, $a \leq A$, $b \leq B$, and $a \leq b$.

First, we prove that the positions (0,0,0,0,0,1),(0,1,1,0,1,1), and (1,1,1,1,1,1) are in the set $\mathcal{P}$. Clearly, the position(0,0,0,0,0,1) is in $\mathcal{P}$ because the first player has to take the only remaining token, and lose the game. Now, consider the position (0,1,1,0,1,1), from which the first player makes a move and comes to either the position (0,0,1,0,1,1) or the position (0,0,0,0,1,1). After any of these two possible moves by the first player, the second player makes a move which leads to the position (0,0,0,0,0,1), forces the first player to take the last token, and wins the game, under misère convention. Similarly, from every possible move by the first player on the position (1,1,1,1,1,1), the second player has a move to either (0,1,1,0,1,1) or (0,0,0,0,0,1), which ensures his win.

It is easy to observe that from any other initial position with $C = 1$, the first player always has a move that leads to either (0, 0, 0, 0, 0, 1) or (0, 1, 1, 0, 1, 1). By playing the corresponding move, the first player forces the game into a losing position for the second player and wins the game.

Now, consider a position (a, b, c, A, B, C) with $C \geq 2$ and $0 \leq a, b, c, A, B \leq 1$. **Claim**: From this position, the next player has a move to one of the positions (0, 0, 0, 0, 0, 1), (0, 1, 1, 0, 1, 1), and (1, 1, 1, 1, 1, 1), and wins the game.

To prove this claim, consider such a position with $a = 0$.

`Case (i)`: $b = 0$. If $c = 1$, then the next player can select the vertex ABC, come to the position (0,0,1,0,0,0), and win the game. If $c = 0$, then the next player can

make a move to the position (0,0,0,0,0,1).
`Case (ii)`: $b = 1$. If $c = 0$, then by selecting the vertex ABC the next player moves to the position (0,1,0,0,0,0). If $c = 1$, the next player has a move by selecting the vertex ABC that leads to the position (0,1,1,0,1,1).

Next, assume that $a = 1$. Then, for $c = 0$ and $c = 1$, the next player selects the vertex ABC and moves to the positions (1,1,0,1,1,0) and (1,1,1,1,1,1), respectively. This completes the proof of the above claim.

Let us now take a position with $C \geq 2$, $a = A$, $b = B$, and $c = C$. After the first player makes a move, the second player should continue playing as he would under normal play, as long as there exists at least one pair of opposite faces where each face has more than one token. As a result, the second player forces the first player to move to a position with exactly one face with more than one token. By applying the above claim, we can now conclude that the second player wins the game, because he is the next player to move.

Finally, it remains to prove that the first player has a winning strategy for any position with $C \geq 2$, where at least one of the conditions $a = A$, $b = B$, or $c = C$ does not hold.

To this end, consider a position not in $\mathcal{P}$, with only $C \geq 2$. Then, by the above claim, the first player wins.

Next, start a position not in $\mathcal{P}$, with more than one face having at least two tokens.
`Case (i)`: $max(b, c) \geq 2$. If $max(b, c) \geq 2$, then at least one of the pairs of opposite faces $(f_a, f_A), (f_b, f_B)$, and (f_c, f_C) has at least two tokens. Then, the first player starts by selecting the vertex ABC and moves to the position (a, b, c, a, b, c) with $c \geq 2$. As argued above, this position favors the previous player (that is, the first player).
`Case (ii)`: $max(b, c) < 2$. If $max(b, c) < 2$, then $B \geq 2$ and $C \geq 2$, since we begin with a position where more than one face has at least two tokens. If, in addition, $a = 1$, then the first player moves by selecting the vertex ABC to the position (1,1,0,1,1,0) or (1,1,1,1,1,1) depending on whether $c = 0$ or $c = 1$, respectively. On the other hand, if $a = 0$ and $b = 0$, the first player has a move at the vertex ABC to the position (0,0,0,0,0,1) when $c = 0$, or to the position (0,0,1,0,0,0) when $c = 1$. Similarly, if $a = 0$ and $b = 1$, the first player has a move at the vertex ABC according to whether $c = 0$ or $c = 1$ to the position (0,1,0,0,0,0) or (0,1,1,0,1,1), respectively.

This shows that the first player wins from the positions that are not in $\mathcal{P}$. □

Analogous to the above result, we can also establish theorems for the $\mathcal{P}$-positions in the CE, CR, and CF games under misère play. Since the proofs are along the same lines, we omit them.

Theorem 7. *In the game CE under misère play, the set of second-player winning* *positions is*

$$\mathcal{P}=\{(a,b,c,A,B,C)\mid C\geq 2,\ a\bigoplus A=b\bigoplus B=c\bigoplus C\}$$
$$\bigsqcup\{(0,0,0,0,0,1),(0,1,1,0,1,1),(0,0,1,1,1,1)\}.$$

The set of $\mathcal{P}$-positions in the game CR under misère play are given in the next theorem.

Theorem 8. *In the game CR under misère play, the set of second-player winning positions is*

$$\mathcal{P}=\{(a,b,c,A,B,C)\mid C\geq 2,\ a+A=b+B=c+C\}$$
$$\bigsqcup\{(0,0,0,0,0,1),(1,1,1,1,1,1)\}.$$

Note that the misère version of the game CF is equivalent to the misère Nim game on six heaps with the number of tokens a,b,c,A,B, and C. As a result, the next theorem follows from Bouton's [3].

Theorem 9. *In the game CF under misère play, the set of second-player winning positions is*

$$\mathcal{P}=\{(a,b,c,A,B,C)\mid C\geq 2\ ,a\bigoplus b\bigoplus c\bigoplus A\bigoplus B\bigoplus C=0\}$$
$$\bigsqcup\{(a,b,c,A,B,C)\mid C<2\ ,a\bigoplus b\bigoplus c\bigoplus A\bigoplus B\bigoplus C=1\}.$$

In the next section, we define the slow version of Nim games on a cube and determine the second player winning positions under normal play.

5. Slow Version of the Nim Games on a Cube under Normal Play

In this section, we consider the slow version of the games CV, CE, CF, and CR. Without loss of generality, we assume that $a+A\leq b+B\leq c+C$, with $a\leq A,\ b\leq B$, and $c\leq C$. We also use the notion of the parity vector for three integers $a+A, b+B$, and $c+C$. Here, $parity(a+A,b+B,c+C)=(e,o,o)$ means that the number $a+A$ is even and the numbers $b+B$ and $c+C$ are odd.

In the slow version of the game CV, on a player's turn, the player selects a vertex and reduces the number of tokens on the faces intersecting at the selected vertex by at most one. However, the player is required to reduce the value on at least one of the intersecting faces.

Theorem 10. *In the slow version of the game CV, the set of second-player winning positions is*

$$\mathcal{P}=\{(a,b,c,A,B,C)\mid parity(a+A,b+B,c+C)=(e,e,e)\}.$$

Proof. First, observe that the terminal position (0,0,0,0,0,0) is in W, where $W = \{(a,b,c,A,B,C) \mid parity(a+A, b+B, c+C) = (e,e,e)\}$. Now, consider a position $p \in W$. Then, by any legal move $p \to p'$, at least one of the numbers $a+A,\ b+B$, or $c+C$ becomes odd, ensuring that $p' \notin W$.

Next, take a position $p = (a,b,c,A,B,C) \notin W$. This implies that at least one of the sums $a+A,\ b+B$, or $c+C$ is odd. Consider the move where the player selects the vertex ABC and removes one token from each face, with the condition that corresponding opposite faces have an odd number of tokens in total. As a result of this move, the subsequent position p' has property that the total number of tokens on all opposite pairs of faces is even, ensuring that $p' \in W$. By Theorem 1, we conclude that $\mathcal{P} = W$. □

In the slow version of the game CE, on a player's turn, the player selects an edge and reduces the number of tokens on the faces intersecting at that edge by at most one. However, the player must reduce the number of tokens of at least one of these intersecting faces.

Theorem 11. *In the slow version of the game CE, the set of second-player winning positions is*

$$\mathcal{P} = \{(a,b,c,A,B,C) \mid parity(a+A, b+B, c+C) = (e,e,e) \text{ or } (o,o,o)\}.$$

Proof. Let $W = \{(a,b,c,A,B,C) \mid parity(a+A, b+B, c+C) = (e,e,e) \text{ or } (o,o,o)\}$. The terminal position of this game is $(0,0,0,0,0,0)$, and it belongs to W.

Consider a position $p = (a,b,c,A,B,C) \in W$. Then, in the next move, the player can change the parity of at most two of the sums from $(a+A, b+B, c+C)$. As a consequence, by any legal move $p \to p' = (a',b',c',A',B',C')$, the resulting parity vector $parity(a'+A', b'+B', c'+C')$ is neither (e,e,e) nor (o,o,o), and so the position p' does not belong to W.

On the other hand, if $p = (a,b,c,A,B,C) \notin W$, then $parity(a+A, b+B, c+C)$ contains both even and odd entries. If exactly one of them is odd, reduce one of the corresponding face values by one and make their sum even. If exactly two of them are odd, select a common edge between these two faces and remove one token from each of them. In both cases, the resulting position has the property that the sum of the tokens on opposite faces is even. Thus, $p' \in W$. □

In the slow version of the game CF, on his turn, the player selects a face and removes exactly one token from it. This game is trivial, and the second-player winning set is as given in [6].

Theorem 12. *In the slow version of the game CF, the set of second-player winning positions is*

$$\mathcal{P} = \{(a,b,c,A,B,C) \mid a+b+c+A+B+C = even\}.$$

Finally, we consider the game CR in the slow version with an initial position $p = (a, b, c, A, B, C)$. To analyze this game, without loss of generality, we assume that $a \leq A$, $b \leq B$, and $c \leq C$. For a given position $p = (a, b, c, A, B, C)$, we define $parity(p) = (q(a), q(b), q(c), q(A), q(B), q(C))$, where each entry is assigned a value of e or o according to the formula:

$$q(m) = \begin{cases} e & \text{if m is even;} \\ o & \text{if m is odd.} \end{cases}$$

Theorem 13. *In the slow version of the game CR, the set of second-player winning positions is*

$$\mathcal{P} = \{p = (a, b, c, A, B, C) \mid parity(p) \in Q\},$$

where $Q = \{(e, e, e, e, e, e), (e, e, e, o, o, o), (o, o, o, e, e, e), (o, o, o, o, o, o)\}$.

Proof. Define the set $W = \{p = (a, b, c, A, B, C) \mid parity(p) \in Q\}$, where the set $Q = \{(e, e, e, e, e, e), (e, e, e, o, o, o), (o, o, o, e, e, e), (o, o, o, o, o, o)\}$.

Clearly, the terminal position (0,0,0,0,0,0) is in W. Start with a position $p \in W$. If the next move consists of reducing 1 or 3 faces by one token in the selected rotation, say $f_a - f_b - f_A - f_B$, then the number of even entries in the resulting position p' is neither 3 nor 6. As a result, p' does not belong to the set W.

Now, consider the move $p = (a, b, c, A, B, C) \to p' = (a-1, b-1, c, A-1, B-1, C)$ by selecting the rotation $f_a - f_b - f_A - f_B$. In this case, both the equations $q(a) = q(c)$ and $q(b) = q(c)$ are violated because $q(c)$ remains the same. Hence $p' \notin W$.

The remaining possible moves are reducing two faces by exactly one token from the selected rotation $f_a - f_b - f_A - f_B$. In this case, at least one of the equations $q(a) = q(c)$ and $q(b) = q(c)$ is not satisfied.
Similarly, any move from $p \in W$ that involves selecting a rotation $f_a - f_c - f_A - f_C$ or $f_b - f_c - f_B - f_C$ results in a position that does not belong to W.

Take a position $p \notin W$. Without loss of generality, assume that $a = min(a, b, c)$.
`Case (i)`: a is odd. If $a = min(a, b, c)$ is odd, then there is a move by selecting the rotation $f_b - f_c - f_B - f_C$, say $p \to p'$ such that $parity(p') = (o, o, o, e, e, e)$ when $q(A) = e$ and $parity(p') = (o, o, o, o, o, o)$ when $q(A) = o$.
`Case (ii)`: a is even. If a is even, then there exists a legal move by selecting the rotation $f_b - f_c - f_B - f_C$, say $p \to p'$ such that $parity(p') = (e, e, e, e, e, e)$ when $q(A) = e$ and $parity(p') = (e, e, e, o, o, o)$ when $q(A) = o$. □

6. Summary and Discussion

We introduced various ways to extend Nim to a cube structure, where tokens are distributed across its faces. Four game variants were introduced: Cube Vertex

(CV), Cube Edge (CE), Cube Face (CF), and Cube Rotation (CR). The rules for removing tokens differ between variants according to the selection of a face, an edge, a vertex, or a rotation of the cube. For each variant, we determined the initial positions from which the second player has a winning strategy both normal play and misère play conventions. In addition, we discussed slow versions of these games and provided the second-player winning positions under normal play.

A natural extension of the game CV to planar graphs involves playing a variant of Nim where tokens are assigned to each face of the graph. In this game, a move consists of selecting a vertex and removing at least one token in total from the faces that share the selected vertex as part of their boundary. Similarly, the game CE can be extended to planar graphs with tokens on the faces, where a move consists of selecting an edge and removing at least one token from the two faces that share the selected edge as their common edge.

It is also natural to consider a variation on a cube where we assign tokens to edges of the cube, and on his turn, the player selects a vertex and removes tokens from the edges that intersect at the chosen vertex. However, this variation has a sub-game structure when zero tokens are assigned to the dashed edges (as shown in Figure 3), that is equivalent to the $CN(6, 2)$, because the sub-game has a cycle with 6 edges, and the player can make a move by removing tokens from any two adjacent edges.

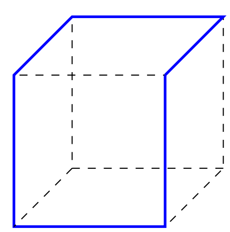

Figure 3: Selecting a vertex game equivalence to $CN(6, 2)$

Another variation with tokens on the edges is to select a face and remove tokens from four edges on its boundary. This variation also gives a sub-game structure equivalent to $CN(6, 2)$ when we assign zero tokens to the dashed edges, as shown in Figure 4.

One may also consider other variations of the game with tokens on the vertices of a cube. On his turn, a player selects an edge of the cube. The player then removes at least one token from the two vertices that are the endpoints of the selected edge. This variation involves a sub-game, which is equivalent to the circular nim $CN(6, 2)$, as shown in Figure 5.

There are other variations of Nim which have $CN(6, 2)$ as a sub-game. It is also interesting to note that $CN(6, 2)$ is the only unsolved circular nim game with

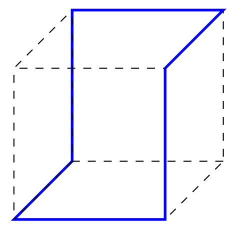

Figure 4: Selecting a face game equivalence to $CN(6,2)$

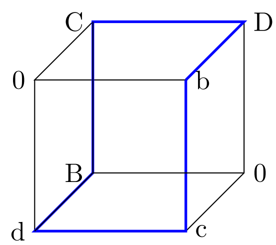

Figure 5: Edge selection game with vertices $a = 0$ and $A = 0$.

n-heaps when $n \leq 6$.

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