GENERALIZATIONS OF TWO-DIMENSIONAL AND THREE-DIMENSIONAL CHOCOLATE BAR GAMES

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Abstract

Chocolate-bar games are variants of Chomp. A two-dimensional (2D) chocolate bar is a rectangular array of squares with some squares removed. There is a bitter square in a given position, and two players take turns breaking the bar along a horizontal or vertical line into two parts and eating the part that does not contain the bitter square. The player who leaves the opponent with a single bitter square wins the game. A three-dimensional (3D) chocolate bar is a generalization of a 2D chocolate bar into three dimensions. In this study, we generalize our previous study to new 2D and 3D chocolate problems. The first generalization is a 2D chocolate bar with height, width, and depth, and we study the condition under which the Grundy number of the chocolate can be expressed as a Nim-sum (BitXor) of the height, width, and depth. We also present 3D chocolate bars with depth. The second generalization is a chocolate bar with more than three dimensions, which is closely related to the game of Nim with a pass move. We modify the standard rules of the game to allow a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position. Once either player uses a pass, it is no longer available. In classical Nim, the introduction of a pass alters the underlying structure of the game, significantly increasing its complexity. For a positive integer s, we can create an $s+1$-dimensional chocolate bar that is mathematically equivalent to s-pile Nim with a pass. Studies on multidimensional chocolate games can present a perspective on the complexity of the game of Nim with a pass.

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1. Introduction

A *two-dimensional (2D) chocolate bar* is a rectangular array of squares from which some of the squares are removed, and a bitter square is included in some parts of the bar. Figures 1, 3, and 4 show examples of 2D chocolate bars. The players take turns cutting the bar in a straight line along the grooves into two parts and eat the part without the bitter square. The player who leaves the opponent with a single bitter square wins the game. A bitter square is printed in black.

A *three-dimensional (3D) chocolate bar* is a 3D array of cubes that includes a bitter cube printed in black. Figures 5, 7, and 8 show examples of 3D chocolate bars. The players take turns cutting the bar on a plane, that is horizontal or vertical along the grooves into two parts, and eat the part without the bitter cube. The player who leaves their opponent with a single bitter cube is the winner. Examples of cut chocolate bars are shown in Figures 9, 10, 11, and 12.

We now present some examples of chocolate bars. Figures 1 - 4 provide examples of 2D chocolate bars and a two-pile Nim, and Figures 5 - 12 provide examples of three-dimensional chocolate bars and a three-pile Nim.

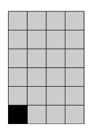
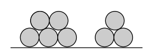


Figure 1.



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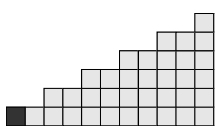

Figure 3.

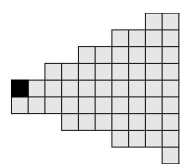

Figure 4.

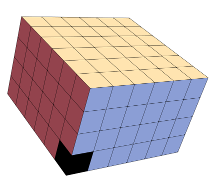

Figure 5.

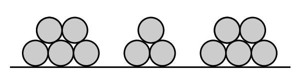

Figure 6.

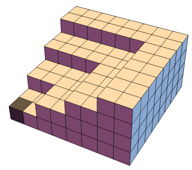

Figure 7.

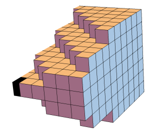

Figure 8.

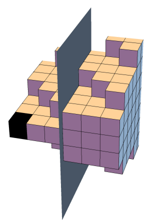

Figure 9.

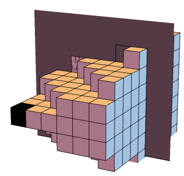

Figure 10.

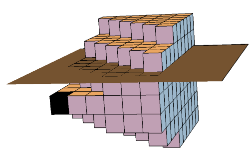

Figure 11.

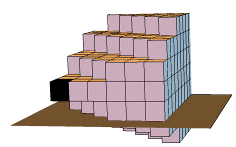

Figure 12.

In the remainder of this article, $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ will denote the set of nonnegative integers and the set of positive integers, respectively.

Definition 1. (i) A function f of $\mathbb{Z}_{\geq 0}$ into itself is *monotonically increasing* if $f(u) \leq f(v)$ for $u, v \in \mathbb{Z}_{\geq 0}$, with $u \leq v$.
(ii) A function F of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ into $\mathbb{Z}_{\geq 0}$ is *monotonically increasing* if $F(w, x) \leq F(u, v)$ for $w, x, u, v \in \mathbb{Z}_{\geq 0}$, with $w \leq u$ and $x \leq v$.

The chocolate bars are defined using monotonically increasing functions.

Definition 2. Let g and h be monotonically increasing functions of (i) in Definition 1. A 2D *chocolate bar* is a rectangular array of squares with some squares removed. A bitter square exists at the position of $(0, 0)$. For $x, y, z \in \mathbb{Z}_{\geq 0}$, the chocolate bar has $z+1$ columns, the height of the i-th column is $\min(g(i), x)+1$, and the depth is $\min(h(i), y)+1$ for $i = 0, 1, \ldots, z$. We denote this chocolate bar by $CB(g, h, x, y, z)$.

Definition 3. Let G and H be monotonically increasing functions of (ii) in Definition 1. A 3D *chocolate bar* comprises a set of cubes of size $1 \times 1 \times 1$ with a bitter cube at $(0, 0, 0)$. For $u, v \in \mathbb{Z}_{\geq 0}$ such that $u \leq w$ and $v \leq x$, the height of the column at position (u, v) is $\min(G(u, v), y) + 1$ and the depth is $\min(H(u, v), z) + 1$. We denote this chocolate bar by $CB(G, H, w, x, y, z)$.

Remark 1. Note that in Definition 2, the first and second coordinates represent the height and depth of the chocolate bar, respectively, whereas in Definition 3, the third and fourth coordinates represent the height and depth, respectively.

Example 1. Let $g(t) = \left\lfloor \frac{t}{2} \right\rfloor$ and $h(t) = \left\lfloor \frac{t}{3} \right\rfloor + 1$, where $= \lfloor \ \rfloor$ denotes the floor function. The chocolate bar $CB(g, h, x, y, z)$ is shown in Figure 13. Figure 14 shows the same chocolate bar with coordinates x, y, z. Note that the functions g and h define the shape of the bar, and the three coordinates x, y, and z denote the number of grooves above, below, and to the right of the bitter square, respectively.

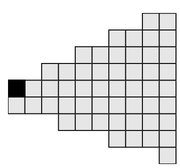

Figure 13.

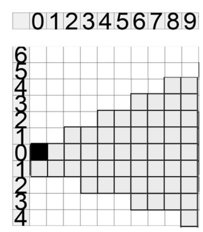

Figure 14: $CB(g, h, 4, 4, 9)$

The original 2D chocolate bar introduced by Robin [8] was rectangular with a bitter square, as shown in Figure 1. Because the horizontal and vertical grooves are independent, an $m \times n$ rectangular chocolate bar is structured similar to the game of Nim, which includes heaps of $m - 1$ and $n - 1$ stones. Therefore, the chocolate bar game (Figure 1) is mathematically equivalent to Nim, which includes heaps of 5 and 3 stones (Figure 2). Because the Grundy number of the Nim game with heaps of $m - 1$ and $n - 1$ stones is $(m - 1) \oplus (n - 1)$, that of the $m \times n$ rectangular bar is $(m - 1) \oplus (n - 1)$.

Robin [8] proposed the use of a cubic chocolate bar, for instance, see Figure 5. It can easily be determined that the 3D chocolate bar in Figure 5 is mathematically equivalent to Nim with heaps of 5, 3, and 5 stones (Figure 6). Hence, the Grundy number of this $6 \times 4 \times 6$ cuboid bar is $5 \oplus 3 \oplus 5$.

Therefore, it is natural to search for a necessary and sufficient condition under which a chocolate bar may have a Grundy number calculated using the Nim-sum of the height and width of 2D chocolate bars, and the Nim-sum of the length, height, and width of 3D chocolate bars. We presented the necessary and sufficient conditions in [3] and [7] when the depth of the chocolate bar was zero.

In this study, chocolate bars are investigated with non-zero depth, as shown in Figures 4 and 8. When the depth is not zero, the situation is significantly different from that without depth. Then, we investigate the chocolate bar with more than three dimensions and apply it to study the game of Nim with a pass move.

Example 2. From a comparison of the chocolate bar in Figure 15 with the three-pile Nim in Figure 16, we can observe some characteristics of the chocolate bar. If the chocolate bar is cut vertically as shown in Figure 15, its width is reduced by two, and its height and depth are reduced by one. In the traditional three-pile Nim, the stones are removed from one pile, whereas in Wythoff's Nim, the number of stones removed from the two piles must be equal. Therefore, the chocolate bar in Figure 15 represents a new type of Nim.

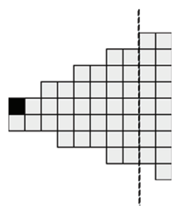

Figure 15.

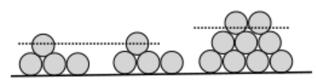

Figure 16.

This study aims to answer the following three questions.

Question 1. *What is a necessary and sufficient condition under which a 2D chocolate bar may have a Grundy number* $(x-1) \oplus (y-1) \oplus (z-1)$*, where* $x, y,$ *and* z *denote the height, depth, and width of the bar, respectively?*

Question 2. *What is a necessary and sufficient condition under which a 3D chocolate bar may have a Grundy number* $(w-1) \oplus (x-1) \oplus (y-1) \oplus (z-1)$*, where* $w, x, y,$ *and* z *denote the length, width, height, and depth of the bar, respectively?*

Question 3. *Can we apply the answer of Question 2 to chocolate bars with more than three dimensions?*

The remainder of this paper is organized as follows. In Section 2, we briefly review the concepts necessary for combinatorial game theory. In Section 3, we present a summary of the research results of 2D and 3D chocolate bar games published in [3] and [7]. In Section 4, we investigate 2D and 3D chocolate bars with depth such as those shown in Figures 4 and 8, and provide an answer to the aforementioned research question. Because we introduce depth into the definition of chocolate bars, we cannot use the proof of the sufficient condition in Lemma 6 of [7]. Therefore, we generalize the proof of Lemma 6 in [7] and apply it to Lemmas 2, 3, 5, and Theorem 4. This generalized proof is simpler than that for Lemma 6 in [7]. To prove the necessary conditions for chocolate bars with depth, we use the results presented in [7].

In Section 5, we investigate multidimensional chocolate bar games in which the chocolate bar dimensions are greater than or equal to three. For multidimensional chocolate bars, the proof of the necessary conditions used in [3] cannot be applied. Therefore, we provide a new proof for Theorem 7. The proof of the necessary condition in Theorem 7 is simpler than that of the necessary condition in [7]. Because the proof of Theorem 7 is a generalization of the proof of the necessary conditions used in [3], the present study is a self-contained presentation of the results, including those of [3] and [7] as special cases.

In Section 6, we apply the theory of multidimensional chocolate bars to that of Nim with a pass move. Because a three-pile Nim with a pass move is a four-dimensional chocolate bar with a fourth dimension for the pass move, the generalization to the case of multidimensional chocolate bars is meaningful.

2. Combinatorial Game Theory Definitions and Theorems

For completeness, we briefly review some of the necessary concepts in combinatorial game theory by referring to [1] and [9].

Definition 4. Let x and y be nonnegative integers. Expressing them in base 2, $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$. We define *Nim-sum*, $x \oplus y$, as follows:

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i,$$

where $w_i = x_i + y_i \pmod 2$.

Because chocolate bar games are impartial games without drawings, only two outcome classes are possible.

Definition 5. (a) A position is referred to as a $\mathcal{P}$-*position* if it is the winning position for the previous player (the player who has just moved), as long as they play correctly at each stage.
(b) A position is referred to as an $\mathcal{N}$-*position* if it is the winning position for the next player, as long as they play correctly at each stage.

Definition 6. The *disjunctive sum* of the two games, denoted by $\mathbf{G}+\mathbf{H}$, is a super game in which a player may move either in $\mathbf{G}$ or $\mathbf{H}$ but not in both.

Definition 7. For any position $\mathbf{p}$ in game $\mathbf{G}$, a set of positions can be reached by a single move in $\mathbf{G}$, which we denote as $move(\mathbf{p})$.

Definition 8. (i) The *minimum excluded value* (*mex*) of a set S of nonnegative integers is the least nonnegative integer that is not in S.
(ii) Let $\mathbf{p}$ be a position in the impartial game. The associated *Grundy number* is denoted by $G(\mathbf{p})$ and is recursively defined by $G(\mathbf{p}) = mex(\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

The next result demonstrates the usefulness of the Sprague-Grundy theory for impartial games.

Theorem 1 ([1])**.** *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be impartial rulesets, and* $G_{\mathbf{G}}$ *and* $G_{\mathbf{H}}$ *be the Grundy numbers of game* $\mathbf{g}$ *played under the rules of* $\mathbf{G}$ *and game* $\mathbf{h}$ *played under those of* $\mathbf{H}$*. Then, we obtain the following:*
(*i*) *For any position* $\mathbf{g}$ *in* $\mathbf{G}$*,* $G_{\mathbf{G}}(\mathbf{g}) = 0$ *if and only if* $\mathbf{g}$ *is the* $\mathcal{P}$*-position.*
(*ii*) *The Grundy number of positions* $\{\mathbf{g}, \mathbf{h}\}$ *in game* $\mathbf{G} + \mathbf{H}$ *is* $G_{\mathbf{G}}(\mathbf{g}) \oplus G_{\mathbf{H}}(\mathbf{h})$.

Using Theorem 1, we can determine the $\mathcal{P}$-position by calculating the Grundy numbers and the $\mathcal{P}$-position of the sum of the two games by calculating the Grundy numbers of the two games.

3. Some Theorems on Two- and Three-Dimensional Chocolate Bars

Here, we describe the cutting of chocolates using coordinates. We use 2D chocolate bars in Example 3 because they are easier to understand. The case of cutting 3D chocolate bars can be understood as a generalization of the case of 2D chocolate bars. Subsequently, some theorems on chocolate bar games are presented. These are Theorems 2 and 3 published by the first author of the present article.

We fix the functions h and g for the chocolate bar $CB(h, g, x, y, z)$ and refer to x, y and z as the coordinates of $CB(h, g, x, y, z)$. For fixed functions h and g, we define $move_{h,g}$ for each position (x, y, z) of the chocolate bar $CB(h, g, x, y, z)$. We set $move_{h,g}(x, y, z)$ to represent the positions of the chocolate bar obtained by cutting the chocolate bar $CB(h, g, x, y, z)$, and $move_{h,g}$ represents a special case of *move* as defined by Definition 7.

Definition 9. For $x, y, z \in \mathbb{Z}_{\geq 0}$, we define

$$\begin{aligned} move_{h,g}(x, y, z) = \{&(u, y, z) : u < x\} \cup \{(x, v, z) : v < y\} \\ &\cup \{(\min(x, h(w)), \min(y, g(w)), w) : w < z\}, \end{aligned}$$

where $u, v, w \in \mathbb{Z}_{\geq 0}$.

Remark 2. For fixed functions h and g, we use $move(x, y, z)$ instead of $move_{h,g}(x, y, z)$.

Example 3. Let $h(t) = \lfloor \frac{t}{2} \rfloor$ and $g(t) = \lfloor \frac{t}{3} \rfloor + 1$, where $= \lfloor \ \rfloor$ denotes the floor function. Here, we present examples of $CB(h, g, x, y, z)$-type chocolate bars. Note that the functions h and g define the shape of the bar and the three coordinates x, y, and z represent the number of grooves above, below, and to the right of the bitter squares, respectively. When we use fixed functions h and g, we represent the chocolate bar positions using the coordinates x, y and z.

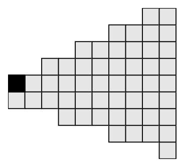

Figure 17.

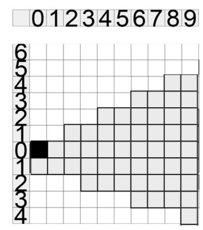

Figure 18: $(4, 4, 9)$

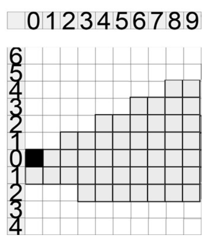

Figure 19: $(4, 2, 9)$

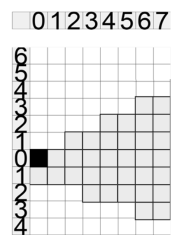

Figure 20: $(3, 3, 7)$

Here, we explain $move_{h,g}$ when $h(t) = \lfloor \frac{t}{2} \rfloor$ and $g(t) = \lfloor \frac{t}{3} \rfloor + 1$. We start with the position $(x, y, z) = (4, 4, 9)$ in Figure 18 and reduce $y = 4$ to $y = 2$. Thus, we obtain $(4, 2, 9) \in move_{h,g}(4, 4, 9)$. See Figure 19. If we start with the position $(x, y, z) = (4, 4, 9)$ in Figure 18 and reduce $z = 9$ to $z = 7$, the x-coordinate (first coordinate) is $\min(4, \lfloor \frac{7}{2} \rfloor) = \min(4, 3) = 3$ and the y-coordinate (second coordinate) is $\min(4, \lfloor \frac{7}{3} \rfloor + 1) = \min(4, 3) = 3$. Then, $(3, 3, 7) \in move_{h,g}(4, 4, 9)$. See Figure 20.

According to Definitions 8 and 9, we define the Grundy number of a 2D chocolate bar in Definition 10.

Definition 10. For $x, y, z \in \mathbb{Z}_{\geq 0}$, we define

$$\begin{aligned} \mathcal{G}_{h,g}(x, y, z) = mex(&\{\mathcal{G}_{h,g}(u, y, z) : u < x, u \in \mathbb{Z}_{\geq 0}\} \\ &\cup \{\mathcal{G}_{h,g}(x, v, z) : v < y, v \in \mathbb{Z}_{\geq 0}\} \\ &\cup \{\mathcal{G}_{h,g}(\min(x, h(w)), \min(y, g(w)), w) : w < z, w \in \mathbb{Z}_{\geq 0}\}). \end{aligned}$$

The *NS property* of the function is defined in Definition 11. Here, NS denotes necessity and sufficiency. We use this NS property to describe the necessary and sufficient condition for a chocolate bar game whose Grundy number equals that of Nim-sum.

Definition 11. Let h be a monotonically increasing function as defined by Definition 1. Function h is said to have the *NS property* if h satisfies the following condition: if

$$\left\lfloor \frac{z}{2^i} \right\rfloor = \left\lfloor \frac{z'}{2^i} \right\rfloor$$

for some z, $z' \in \mathbb{Z}_{\geq 0}$, and a positive integer i, then

$$\left\lfloor \frac{h(z)}{2^{i-1}} \right\rfloor = \left\lfloor \frac{h(z')}{2^{i-1}} \right\rfloor.$$

Lemma 1. *Let $z \in \mathbb{Z}_{\geq 0}$. Let $h(x)$ be a function defined for $x \in \mathbb{Z}_{\geq 0}$ such that $x \leq z$. Suppose that h possesses the NS property in Definition 11. If we define $\hat{h}(x)$ as*

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x \leq z, \\ h(z) & \text{if } x > z, \end{cases}$$

then the function $\hat{h}$ has the NS property.

Proof. This is directly derived from Definition 11. □

Theorem 2 ([7])**.** *Let f be a monotonically increasing function and 0 be a function that is constantly 0. Let $\mathcal{G}_{f,0}$ be the Grundy number of $CB(f, 0, x, 0, z)$. Then, f has the NS property according to Definition 11 if and only if $\mathcal{G}_{f,0}(\{x, 0, z\}) = x \oplus z$.*

In Definition 1.2 of [7], a chocolate bar is described by one function and two coordinates, where the first coordinate represents the height. In this study, according to Definition 2, the same chocolate bar can be defined using two functions and three coordinates, where one of the functions is constantly 0, the first coordinate represents the height, the second coordinate represents the depth 0, and the third coordinate represents the width.

Theorem 3 ([3])**.** *Let $F(w, x)$ be a monotonically increasing function and 0 be a function that is constantly 0, and let $g_n(x) = F(n, x)$ and $h_m(w) = F(w, m)$ for $n, m \in \mathbb{Z}_{\geq 0}$. Then, g_n and h_m satisfy the NS property for any fixed $n, m \in \mathbb{Z}_{\geq 0}$ if and only if the Grundy number of chocolate bars $CB(F, 0, w, x, y, 0)$ is*

$$\mathcal{G}_{F,0}(\{w, x, y, 0\}) = w \oplus x \oplus y.$$

Note that in Definition 4.2 of [3], a chocolate bar is denoted by $CB(F, x, y, z)$ with function F and three coordinates x, y and z, where the second coordinate represents the height. In this study, according to Definition 3, the same chocolate bar can be defined by two functions and four coordinates, where the second function is constantly 0, the fourth coordinate for depth is 0, and the third coordinate denotes the height.

The sufficient part in Theorem 4 is a generalization of the sufficient part in Theorems 2 and 3, and the necessary part of Theorem 7 is a generalization of the necessary part in Theorems 2 and 3. Because Theorems 4 and 7 are proved in the present article without using the results of other articles, all the results in the present article are self-contained.

4. Grundy Numbers of Chocolate Bars with Depth

In a previous study, the first author of the present article studied a chocolate bar similar to that shown in Figure 3. See [7]. However, in this study, we investigate chocolate in terms of height and depth, as shown in Figure 4, which is a natural generalization of the chocolate in Figure 3. We use the functions $h(z)$ and $g(z)$ to describe the height and depth of a bar of two-dimensional chocolate, respectively, where z denotes its width. We assume that functions $h(z)$ and $g(z)$ have the NS property, and prove Theorem 4, which is a generalization of Theorem 2. To prove Theorem 4, we generalize the method of proof used for Theorem 2, which requires some lemmas. We must also generalize the proof method used in Theorem 3 to prove Theorem 6.

Lemma 2. *(i) If*

$$\left\lfloor \frac{a}{2^k} \right\rfloor = \left\lfloor \frac{b}{2^k} \right\rfloor \tag{1}$$

for $a, b, k \in \mathbb{Z}_{\geq 0}$, then, for any $c \in \mathbb{Z}_{\geq 0}$,

$$\left\lfloor \frac{\min(a,c)}{2^k} \right\rfloor = \left\lfloor \frac{\min(b,c))}{2^k} \right\rfloor. \tag{2}$$

(ii) Let h have the NS property, and if

$$\left\lfloor \frac{z}{2^k} \right\rfloor = \left\lfloor \frac{z'}{2^k} \right\rfloor \tag{3}$$

for $z, z' \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$, then, for any $y \in \mathbb{Z}_{\geq 0}$, we obtain

$$\left\lfloor \frac{\min(h(z),y)}{2^{k-1}} \right\rfloor = \left\lfloor \frac{\min(h(z'),y)}{2^{k-1}} \right\rfloor. \tag{4}$$

Proof. (i) It is sufficient to prove the result in the case where

$$a < c < b. \tag{5}$$

Let

$$a = \sum_{i=0}^{n} a_i 2^i. \tag{6}$$

Then, from Equation (1) and Inequality (5), we obtain

$$b = \sum_{i=k}^{n} a_i 2^i + b'$$

and

$$c = \sum_{i=k}^{n} a_i 2^i + c' \tag{7}$$

with $\sum_{i=0}^{k-1} a_i 2^i < c' < b' < 2^{k-1}$. As $\min(b, c) = c$ and $\min(a, c) = a$, from Equations (6) and (7), we obtain Equation (2).
(ii) Because h has the NS property, from Equation (3) and Definition 11, we have

$$\left\lfloor \frac{h(z)}{2^{k-1}} \right\rfloor = \left\lfloor \frac{h(z')}{2^{k-1}} \right\rfloor.$$

Subsequently, Equation (4) is directly derived from (i). □

Lemma 3. *Let g and h be functions that satisfy the NS property. Suppose that*

$$x \leq g(z) \tag{8}$$

and

$$y \leq h(z). \tag{9}$$

Then, we obtain the following.
(a) *If*

$$c < x \oplus z$$

for $c, x, z \in \mathbb{Z}_{\geq 0}$, then at least one of the following statements is true:
(i) *$c = u \oplus z$ for some $u \in \mathbb{Z}_{\geq 0}$ such that $u < x$;*
(ii) *$c = \min(g(w), x) \oplus w$ for some $w \in \mathbb{Z}_{\geq 0}$ such that $w < z$.*
(b) *If*

$$c < x \oplus y \oplus z \tag{10}$$

for $c, x, y, z \in \mathbb{Z}_{\geq 0}$, then at least one of the following statements is true:
(i) *$c = u \oplus y \oplus z$ for some $u \in \mathbb{Z}_{\geq 0}$ such that $u < x$;*
(ii) *$c = x \oplus v \oplus z$ for some $v \in \mathbb{Z}_{\geq 0}$ such that $v < y$;*
(iii) *$c = \min(g(w), x) \oplus \min(h(w), y) \oplus w$ for some $w \in \mathbb{Z}_{\geq 0}$ such that $w < z$.*

Proof. It is sufficient to prove (b). We express nonnegative integers c, x, y and z with base 2 as $c = \sum_{i=0}^{n} c_i 2^i$, $x = \sum_{i=0}^{n} x_i 2^i$, $y = \sum_{i=0}^{n} y_i 2^i$, and $z = \sum_{i=0}^{n} z_i 2^i$, where $c_i, x_i, y_i, z_i \in \{0, 1\}$ and $n \in \mathbb{Z}_{\geq 0}$. From Inequality (10), there exists $s \in \mathbb{Z}_{\geq 0}$ that satisfies

$$s \leq n,$$

$$c_i = x_i + y_i + z_i \pmod 2 \tag{11}$$

for $i = s+1, s+2, \ldots, n$ and

$$c_s = 0 < 1 = x_s + y_s + z_s \pmod 2. \tag{12}$$

Thus, we have three cases.

`Case 1`: Suppose that $x_s = 1$. Then, let

$$x'_i = c_i + x_i + z_i \pmod 2$$

for $i = s, s-1, \ldots, 0$ and $u = \sum_{i=s+1}^{n} x_i 2^i + \sum_{i=0}^{s} x'_i 2^i$. Then, $c = u \oplus x \oplus z$ and we obtain (i).

`Case 2`: Assume that $y_s = 1$. Then, by using a method similar to that used in (i), we obtain (ii).

`Case 3`: Suppose that $x_s = 0$, $y_s = 0$, and $z_s = 1$. Let

$$z'_s = c_s + x_s + y_s = 0 \pmod 2 \tag{13}$$

and $z^{(s)} = \sum_{i=s+1}^{n} z_i 2^i + z'_s 2^s$. Hereafter, we define z'_j for $j = s, s-1, \ldots, 0$ to obtain $z^{(j)}$ for $j = s, s-1, \ldots, 0$ stepwise. Because

$$\left\lfloor \frac{z}{2^{s+1}} \right\rfloor = \left\lfloor \frac{z^{(s)}}{2^{s+1}} \right\rfloor,$$

using Inequalities (8), (9), and (ii) in Lemma 2, we have

$$\left\lfloor \frac{x}{2^s} \right\rfloor = \left\lfloor \frac{\min(g(z), x)}{2^s} \right\rfloor = \left\lfloor \frac{\min(g(z^{(s)}), x)}{2^s} \right\rfloor \tag{14}$$

and

$$\left\lfloor \frac{y}{2^s} \right\rfloor = \left\lfloor \frac{\min(h(z), y)}{2^s} \right\rfloor = \left\lfloor \frac{\min(h(z^{(s)}), y)}{2^s} \right\rfloor. \tag{15}$$

By Equation (14), x and $\min(g(z^{(s)}), x)$ have the same place value of 2^k for $k = s, s+1, \ldots, n$, and by Equation (15), y and $\min(g(z^{(s)}), y)$ have the same place value of 2^k for $k = s, s+1, \ldots, n$.

Therefore, there exist $x'_{s-1}, y'_{s-1} \in \{0, 1\}$, a_{s-1} and b_{s-1} such that $0 \leq a_{s-1}, b_{s-1} < 2^{s-1}$,

$$\min(g(z^{(s)}), x) = \sum_{i=s}^{n} x_i 2^i + x'_{s-1} 2^{s-1} + a_{s-1}$$

and

$$\min(h(z^{(s)}), y) = \sum_{i=s}^{n} y_i 2^i + y'_{s-1} 2^{s-1} + b_{s-1}.$$

Therefore, from Equation (11), Inequality (12), and Equation (13),

$$c_i = \min(g(z^{(s)}), x)_i + \min(h(z^{(s)}), y)_i + (z^{(s)})_i \tag{16}$$

for $i = s, s+1, \ldots, n$. We use mathematical induction. Suppose there exists $j \in \mathbb{N}$ such that $j < s$ and

$$c_i = \min(g(z^{(j)}), x)_i + \min(h(z^{(j)}), y)_i + (z^{(j)})_i$$

for $i = j, j+1, \ldots, n$. By using a method similar to that used to obtain Equation (16) from Equation (13), we obtain $z^{(j-1)}$ such that

$$c_i = \min(g(z^{(j-1)}), x)_i + \min(h(z^{(j-1)}), y)_i + (z^{(j)})_i$$

for $i = j-1, j, \ldots, n$.

We continue this process until we obtain $z^{(0)}$ such that

$$c_i = \min(g(z^{(0)}), x)_i + \min(h(z^{(0)}), y)_i + (z^{(0)})_i$$

for $i = 0, 1, \ldots, n$, and we obtain (iii) for $w = z^{(0)}$. □

Lemma 4. *Let $m \in \mathbb{N}$, and for $t = 1, 2, \ldots, m$, let f_t be functions that satisfy the NS property. Suppose that*

$$x_t \leq f_t(z),$$

and

$$c < x_1 \oplus x_2 \oplus \cdots \oplus x_m \oplus z$$

for $c, x_1, x_2, \ldots, x_m \in \mathbb{Z}_{\geq 0}$. Then, at least one of the following statements is true:
(i) $c = \bigoplus_{1 \leq t \leq m, t \neq i} x_t \oplus u \oplus z$ for some $u \in \mathbb{Z}_{\geq 0}$ such that $u < x_i$;
(ii) $c = \bigoplus_{t=1}^{m}, \min(x_t, f_t(w)) \oplus w$ for some $w \in \mathbb{Z}_{\geq 0}$ such that $w < z$.

Proof. We can prove this lemma by demonstrating the generalization of Lemma 3. We must use functions f_t for $t = 1, 2, \ldots, m$ instead of the two functions g and h; however, we omit the proof of this lemma because the method of the proof is practically apparent, and the use of functions f_t for $t = 1, 2, \ldots, m$ makes the description of the proof excessively complicated. □

Lemma 5. *Let g and h be functions that satisfy the NS property. Suppose that*

$$x \leq g(z) \tag{17}$$

and

$$y \leq h(z). \tag{18}$$

Then, we obtain equations

$$\begin{aligned} &\{x \oplus y \oplus w : w \in \mathbb{Z}_{\geq 0} \textit{ and } w < z\} \\ &\quad = \{\min(g(w'), x) \oplus \min(h(w'), y) \oplus w' : w' \in \mathbb{Z}_{\geq 0} \textit{ and } w' < z\} \end{aligned} \tag{19}$$

and

$$\{x \oplus w : w \in \mathbb{Z}_{\geq 0} \textit{ and } w < z\} = \{\min(g(w'), x) \oplus w' : w' \in \mathbb{Z}_{\geq 0} \textit{ and } w' < z\}. \tag{20}$$

Proof. First, we prove Equation (19). Let $w \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$ such that $w < z$ and

$$x, y, z < 2^m. \tag{21}$$

If 2^m does not belong to the domains of functions g and h, we can make the domains of each function using Lemma 1 sufficiently large to include 2^m. From Inequality (21), we obtain

$$x \oplus y \oplus w < x \oplus y \oplus 2^m.$$

From Lemma 3, we obtain

$$x \oplus y \oplus w = u \oplus y \oplus 2^m \tag{22}$$

for u such that $u < x$ or

$$x \oplus y \oplus w = x \oplus v \oplus 2^m \tag{23}$$

for v such that $v < y$ or

$$x \oplus y \oplus w = \min(x, g(w')) \oplus \min(y, h(w')) \oplus w' \tag{24}$$

for w' such that $w' < 2^m$. Equations (22) and (23) contradict Inequality (21). Therefore, we have Equation (24). If $w' \geq z$, then from Inequalities (17) and (18), $\min(x, g(w')) = x$ and $\min(y, h(w')) = y$. Subsequently, from Equation (24), we obtain

$$x \oplus y \oplus w = x \oplus y \oplus w',$$

which contradicts the assumption that $w < z \leq w'$. Therefore, we obtain $w' < z$ and

$$\{x \oplus y \oplus w : w < z\} \subset \{\min(g(w'), x) \oplus \min(h(w'), y) \oplus w' : w' < z\}. \tag{25}$$

The number of elements in $\{x \oplus y \oplus w : w < z\}$ is z and those in $\{\min(g(w'), x) \oplus \min(h(w'), y) \oplus w' : w < z\}$ are less than or equal to z. Therefore, from Relation (25), we obtain Equation (19).

Next, we prove Equation (20). Let $y = 0$ and $h(t) = 0$ for any $t \in \mathbb{Z}_{\geq 0}$ in Equation (19). Then, we have Equation (20). $\square$

Lemma 6. *Let $m \in \mathbb{N}$, and for $t = 1, 2, \ldots, m$, let f_t be functions that satisfy the NS property. Suppose that*

$$x_t \leq f_t(z).$$

Then,

$$\{\bigoplus_{t=1 j \neq i}^{m} x_j \oplus w : w < z\} = \{\bigoplus_{t=1}^{m} \min(x_t, f_t(w)) \oplus w : w < z\}.$$

Proof. We can prove this lemma by a proof that is a generalization of the proof of Lemma 5 using functions f_t for $t = 1, 2, \ldots, m$ instead of two functions g and h. The proof is straightforward; therefore, details are omitted. $\square$

Theorem 4. *Let g and h be monotonically increasing functions, and $\mathcal{G}_{g,h}(x, y, z)$ be the Grundy number of $CB(g, h, x, y, z)$. Then, g and h satisfy the NS property, if and only if $\mathcal{G}_{g,h}(x, y, z) = x \oplus y \oplus z$.*

Proof. Let $x, y, z \in \mathbb{Z}_{\geq 0}$ such that $x \leq g(z)$ and $y \leq h(z)$. We assume that g and h satisfy the NS property and prove that $\mathcal{G}_{g,h}(x, y, z) = x \oplus y \oplus z$ through mathematical induction, and assume that $\mathcal{G}_{g,h}(u, v, w) = u \oplus v \oplus w$ for any u, v and w such that $u \leq x$, $v \leq y$, $w \leq z$, $u + v + w < x + y + z$, $u \leq g(w)$, and $v \leq h(w)$. From the definition of the Grundy number in Definition 8,

$$\begin{aligned} \mathcal{G}_{g,h}(x, y, z) &= mex(\{\mathcal{G}_{g,h}(u, v, w) : (u, v, w) \in move_{g,h}(x, y, z)\}) \\ &= mex(\{\mathcal{G}_{g,h}(u, y, z) : u < x\} && (26) \\ &\quad \cup \{\mathcal{G}_{g,h}(x, v, z) : v < y\} && (27) \\ &\quad \cup \{\mathcal{G}_{g,h}(\min(g(w), x), \min(h(w), y), w) : w < z\}). && (28) \end{aligned}$$

Based on the hypothesis of mathematical induction, we obtain

$$\{\mathcal{G}_{g,h}(u, y, z) : u < x\} = \{u \oplus y \oplus z : u < x\} \tag{29}$$

and

$$\{\mathcal{G}_{g,h}(x, v, z) : v < y\} = \{x \oplus v \oplus z : v < y\}. \tag{30}$$

From the mathematical induction hypothesis and Lemma 5

$$\begin{aligned}&\{\mathcal{G}_{g,h}(\min(g(w),x),\min(h(w),y),w) : w < z\}\\ &\quad = \{\min(g(w),x) \oplus \min(h(w),y) \oplus w : w < z\}\\ &\quad = \{x \oplus y \oplus w : w < z\}. \end{aligned} \tag{31}$$

From Relations (26), (27), and (28) and Equations (29), (30), and (31), we obtain:

$$\begin{aligned}&\mathcal{G}_{g,h}(x,y,z)\\ &\quad = mex(\{u \oplus y \oplus z : u < x\} \cup \{x \oplus v \oplus z : v < y\} \cup \{x \oplus y \oplus w : w < z\})\\ &\quad = x \oplus y \oplus z.\end{aligned}$$

Conversely, we assume that $\mathcal{G}_{g,h}(x,y,z) = x \oplus y \oplus z$. Subsequently, $\mathcal{G}_{g,h}(x,0,z) = x \oplus 0 \oplus z = x \oplus z$. Chocolate bar $CB(g,h,x,0,z)$ is the same as chocolate bar $CB(g,0,x,0,z)$, and by Theorem 2, g satisfies the NS property. Similarly, by using $\mathcal{G}_{g,h}(0,y,z) = 0 \oplus y \oplus z = y \oplus z$, we prove that h has the NS property. □

Lemma 7. *Let $m \in \mathbb{N}$ and*

$$f(x) = \left\lfloor \frac{x}{2m} \right\rfloor.$$

Then, f satisfies the NS property.

For the proof, see Lemma 4 on page 10 of [7].

Example 4. Let $h(t) = \lfloor \frac{t}{2} \rfloor$ and $g(t) = \lfloor \frac{t}{4} \rfloor$, where $= \lfloor \quad \rfloor$ denotes the floor function. Here, we present examples of the Grundy numbers of $CB(h,g,x,y,z)$-type chocolate bars. Because we use fixed functions g and h, we represent the chocolate bar positions using the coordinates x, y and z. By Lemma 7, g and h satisfy the NS property. Hence, by Theorem 4, for chocolate bars in Figures 21 and 22, we have $\mathcal{G}_{g,h}(4,2,9) = 4 \oplus 2 \oplus 9 = 15$ and $\mathcal{G}_{g,h}(2,2,9) = 2 \oplus 2 \oplus 9 = 9$.

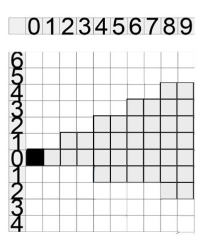

Figure 21: $(4,2,9)$

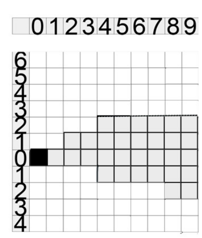

Figure 22: $(2,2,9)$

Next, we define a chocolate bar using the five coordinates in Definition 12. Figure 24 illustrates an example of this type of chocolate.

Definition 12. Let f_1, f_2, f_3 and f_4 be monotonically increasing functions in (i) of Definition 1. We define a chocolate bar with five coordinates: x_1, x_2, x_3, x_4, and z. See the coordinates system in Figure 23. A 3D chocolate bar comprises a set of cubes with a bitter cube at $(0,0,0,0,0)$. The shape of the chocolate along the x_i-coordinates is determined by $\min(f_i(z), x_i) + 1$. We denote this chocolate by $CB(f_1, f_2, f_3, f_4, x_1, x_2, x_3, x_4, z)$.

To describe the shape of the chocolate in Figure 24, five coordinates are required, as shown in Figure 23. Using the bitter cube as the origin coordinate, x_1, x_2, x_3, x_4, and z denote the width to the right of the origin, height, width to the left of the origin, depth, and length of the chocolate, respectively.

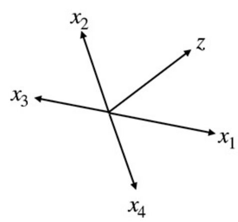

Figure 23: five coordinates

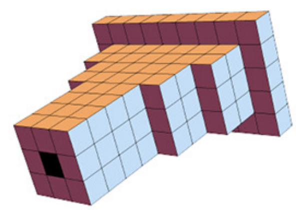

Figure 24: A chocolate bar with five coordinates

Theorem 5. *For $t = 1, 2, 3, 4$, let f_t be a monotonically increasing function in Definition 1. Let $\mathcal{G}_{f_1,f_2,f_3,f_4}(x_1, x_2, x_3, x_4, z)$ be the Grundy number of $CB(f_1, f_2, f_3, f_4, x_1, x_2, x_3, x_4, z)$. Then, f_1, f_2, f_3 and f_4 satisfy the NS property if and only if $\mathcal{G}_{f_1,f_2,f_3,f_4}(x_1, x_2, x_3, x_4, z) = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus z$.*

Proof. Using Lemma 6, we can prove this theorem in a manner similar to that used for Theorem 4. The proof is straightforward; therefore, the details are omitted. □

The first author of the present article studied a chocolate bar similar to that shown in Figure 7 in [3]. However, the theory in [3] cannot address the chocolate in Figure 8, which is a natural generalization of the chocolate in Figure 7. Therefore, Theorem 6 is required.

Next, we study a 3D chocolate bar with depth $CB(F, G, w, x, y, z)$. We need to define $move_{F,G}$. We set $move_{F,G}(w, x, y, z)$ as the set containing all positions that can be reached from position (w, x, y, z) in one direct step.

Definition 13. For $w, x, y, z \in \mathbb{Z}_{\geq 0}$, we define:

$$\begin{aligned} move_{F,G}(w,x,y,z) = &\{(w', x, \min(F(w', x), y), \min(G(w', x), z)) : w' < w\} \\ &\cup \{(w, u, \min(F(w, u), y), \min(G(w, u), z)) : u < x\} \\ &\cup \{(w, x, v, z)\} : v < y\} \cup \{(w, x, y, z')\} : z' < z\}, \end{aligned}$$

where $w', u, v, z' \in \mathbb{Z}_{\geq 0}$.

Theorem 6. *Let* $\overline{g}_x(w) = F(w,x)$, $\overline{h}_w(x) = F(w,x)$, $\underline{g}_x(w) = G(w,x)$, $\underline{h}_w(x) = G(w,x)$. *We suppose that* $\mathcal{G}_{F,G}(w,x,y,z)$ *is the Grundy number of* $CB(F,G,w,x,y,z)$. *Then,*

$$\mathcal{G}_{F,G}(w,x,y,z) = w \oplus x \oplus y \oplus z \tag{32}$$

if and only if $\overline{g}_x(w), \overline{h}_w(x), \underline{g}_x(w)$, *and* $\underline{h}_w(x)$ *satisfy the NS property.*

Proof. We assume that functions $\overline{g}_x(w), \overline{h}_w(x), \underline{g}_x(w)$, and $\underline{h}_w(x)$ satisfy the NS property. We prove Equation (32) through mathematical induction. By the mathematical induction hypothesis,

$$\begin{aligned}
\mathcal{G}_{F,G}(w,x,y,z) =& mex(\{\mathcal{G}_{F,G}(w',u,v,z') : (w',u,v,z') \in move(w,x,y,z)\}) \\
=& mex(\{\mathcal{G}_{F,G}(w',x,\min(F(w',x),y),\min(G(w',x),z)) : w' < w\} \\
& \cup \{\mathcal{G}_{F,G}(w,x',\min(F(w,x'),y),\min(G(w,x'),z)) : x' < x\} \\
& \cup \{\mathcal{G}_{F,G}(w,x,y',z) : y' < y\} \\
& \cup \{\mathcal{G}_{F,G}(w,x,y,z') : z' < z\}) \\
=& mex(\{w' \oplus x \oplus \min(\overline{g}_x(w'),y) \oplus \min(\underline{g}_x(w'),z) : w' < w\} \\
& \cup \{w \oplus x' \oplus \min(\overline{h}_w(x'),y) \oplus \min(\underline{h}_w(x'),z)) : x' < x\} \\
& \cup \{w \oplus x \oplus y',z) : y' < y\} \\
& \cup \{w \oplus x \oplus y \oplus z') : z' < z\}).
\end{aligned} \tag{33}$$

By Lemma 5,

$$\begin{aligned}
&\{w' \oplus x \oplus \min(\overline{g}_x(w'),y) \oplus \min(\underline{g}_x(w'),z) : w' < w\} \\
&\quad = \{w' \oplus x \oplus y \oplus z : w' < w\}
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&\{w \oplus x' \oplus \min(\overline{h}_w(x'),y) \oplus \min(\underline{h}_w(x'),z)) : x' < x\} \\
&\quad = \{w \oplus x' \oplus y \oplus z : x' < x\}.
\end{aligned} \tag{35}$$

By Equations (33), (34), and (35), we obtain Equation (32).

Next, we assume Equation (32). Then,

$$\mathcal{G}_{F,G}(w,x,y,0) = w \oplus x \oplus y \oplus 0.$$

Chocolate bar $CB(F,G,w,x,y,0)$ is the same as chocolate bar $CB(F,0,w,x,y,0)$, and from Theorem 3, $\overline{g}_x(w)$ and $\overline{h}_w(x)$ satisfy the NS property. Similarly, by

$$\mathcal{G}_{F,G}(w,x,0,z) = w \oplus x \oplus 0 \oplus z,$$

we prove that $\underline{g}_x(w)$ and $\underline{h}_w(x)$ satisfy the NS property. □

5. Multidimensional Chocolate Bar

We investigate a multidimensional chocolate bar game in which the chocolate bar dimensions are greater than or equal to three. In this section, we assume that s is a fixed positive integer.

Definition 14. Suppose that $F(x_1, x_2, \dots, x_s)$ is a function with values in $\mathbb{Z}_{\geq 0}$ defined for variables $x_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \dots, s$. A function F is said to be *monotonically increasing* if $F(x_1, x_2, \dots, x_s) \leq F(x'_1, x'_2, \dots, x'_s)$ for $x_i, x'_i \in \mathbb{Z}_{\geq 0}$, with $x_i \leq x'_i$ for $i = 1, 2, \dots, s$.

In this section, we assume that F is a monotonically increasing function. Here, we define a multi-dimensional chocolate bar. This type of chocolate bar can be applied to the game of Nim with a pass move, as described in Section 6.

Definition 15. Let $x_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, \dots, s$ and $y \in \mathbb{Z}_{\geq 0}$. The $(s+1)$-*dimensional chocolate bar* comprises a set of $1 \times 1 \times 1 \times \cdots \times 1$-sized $(s+1)$-dimensional cubes. For $u_i \in \mathbb{Z}_{\geq 0}$, such that $u_i \leq x_i$, the length of the $(s+1)$-th dimension of the column at position $(u_1, u_2, \dots, u_s)$ is $\min(F(u_1, u_2, \dots, u_s), y) + 1$. A bitter cube exists at the position $(0, 0, \dots, 0)$. We denote this chocolate bar by $CB(F, x_1, x_2, \dots, x_s, y)$.

Definition 16. We define an $(s+1)$-*dimensional chocolate bar game*. The players take turns cutting the bar on a hyperplane that is vertical to the x_i-axis for some $i \in \{1, 2, \dots, s\}$ and eat the broken piece without the bitter cube. The player who manages to leave the opponent with a bitter cube is the winner.

Lemma 8. *Let $i, z, z' \in \mathbb{Z}_{\geq 0}$. We obtain (a) and (b).*
(a) For $z < z'$,

$$\left\lfloor \frac{z}{2^i} \right\rfloor = \left\lfloor \frac{z'}{2^i} \right\rfloor,$$

if and only if there exists $d \in \mathbb{Z}_{\geq 0}$ such that

$$d \times 2^i \leq z < z' < (d+1) \times 2^i.$$

(b) Suppose that

$$\left\lfloor \frac{z}{2^i} \right\rfloor < \left\lfloor \frac{z'}{2^i} \right\rfloor. \tag{36}$$

Then, there exist $c, s, t \in \mathbb{Z}_{\geq 0}$ such that $s \geq i$, $0 \leq t < 2^s$, and

$$z = c \times 2^{s+1} + t < c \times 2^{s+1} + 2^s \leq z'. \tag{37}$$

Proof. Let $z = \sum_{i=0}^{n} z_i 2^i$ and $z' = \sum_{i=0}^{n} z'_i 2^i$. The relationship in (a) follows directly from the definition of the floor function. Next, we prove (b). According to Inequality (36), two cases exist.

`Case 1`: Suppose that $z < 2^n \leq z'$. Let $c = 0$ and $t = z$. Subsequently, we obtain the following Inequality (37).

`Case 2`: Suppose there exists $s \in \mathbb{Z}_{\geq 0}$ such that $s \geq i$ and $z_k = z'_k$ for $k = n, n-1, ..., s+1$ and $z_s = 0 < 1 = z'_s$. Then, there exist $c, t \in Z_{\geq 0}$ that satisfy Inequality (37). □

Next, we define $move_F(x_1, x_2, \ldots, x_s, y)$ as in Definition 17. We set $move_F(x_1, x_2, \ldots, x_s, y)$ as the set containing all positions that can be reached from position $(x_1, x_2, \ldots, x_s, y)$ in a single step (directly).

Definition 17. For $x_1, x_2, \ldots, x_s, y \in \mathbb{Z}_{\geq 0}$, we define

$$\begin{aligned} &move_F(x_1, x_2, \ldots, x_s, y) \\ &\quad = \cup_{i=1}^{n} \{(x_1, x_2, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_s, \\ &\qquad \min(F(x_1, x_2, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_s), y)) : u < x_i\} \\ &\qquad \cup \{(x_1, x_2, \ldots, x_s, v) : v < y\}, \text{where } u, v \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Remark 3. For a fixed function F, we use $move(x_1, x_2, \ldots, x_s, y)$ instead of $move_F(x_1, x_2, \ldots, x_s, y)$.

Definition 18. For a function F and $k \in \{1, 2, 3, \ldots, s\}$, we fix variables $x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s$, and define

$$F_{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s}(x_k) = F(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_s)$$

for $x_k \in \mathbb{Z}_{\geq 0}$.

Theorem 7. *The function $F_{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s}$ satisfies the NS property for any fixed variables $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s \in \mathbb{Z}_{\geq 0}$ if and only if the Grundy number of chocolate bar $CB(F, x_1, x_2, \ldots, x_s, y)$ is*

$$\mathcal{G}(\{x_1, x_2, \ldots, x_s, y\}) = x_1 \oplus x_2 \oplus \cdots \oplus x_s \oplus y. \tag{38}$$

Proof. Let $g_k(x_k) = F_{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s}(x_k)$. We assume that functions $g_1, g_2, \ldots, g_{s-1}$ and g_s satisfy the NS property, and we prove Equation (38) through mathematical induction. By the mathematical induction hypothesis,

$$\begin{aligned} &\mathcal{G}(x_1, x_2, \ldots, x_s, y) \\ &\quad = mex(\{\mathcal{G}(u_1, u_2, \ldots, u_s) : (u_1, u_2, \ldots, u_s) \in move_F(x_1, x_2, \ldots, x_s, y)\}) \\ &\quad = mex(\cup_{k=1}^{s} \{\mathcal{G}(x_1, x_2, \ldots, x_{k-1}, w, x_{k+1}, \ldots, x_s, \\ &\qquad\quad \min(F(x_1, x_2, \ldots, x_{k-1}, w, x_{k+1}, \ldots, x_s), y)) : w < x_k\} \\ &\qquad \cup \{\mathcal{G}(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_s, v) : v < y\}) \\ &\quad = mex(\cup_{k=1}^{s} \{x_1 \oplus \cdots x_{k-1} \oplus w \oplus x_{k+1} \oplus \cdots \oplus x_s \oplus \\ &\qquad\quad \min(F(x_1, x_2, \ldots, x_{k-1}, w, x_{k+1}, \ldots, x_s), y) : w < x_k\} \\ &\qquad \cup \{x_1 \oplus \cdots x_{k-1} \oplus x_k \oplus x_{k+1} \oplus w : w < y\}). \end{aligned} \tag{39}$$

Because g_k satisfies the NS property, using (ii) of Lemma 5, we obtain

$$\begin{aligned}&\{x_1 \oplus \cdots x_{k-1} \oplus w \oplus x_{k+1} \oplus \cdots \oplus x_s \oplus \\ &\quad \min(F(x_1, x_2, \ldots, x_{k-1}, w, x_{k+1}, \ldots, x_s), y) : w < x_k\} \\ &\quad = \{x_1 \oplus \cdots x_{k-1} \oplus w \oplus x_{k+1} \oplus \cdots \oplus x_s \oplus \min(g_k(w), y) : w < x_k\} \\ &\quad = \{x_1 \oplus \cdots x_{k-1} \oplus w \oplus x_{k+1} \oplus \cdots \oplus x_s \oplus y : w < x_k\}. \qquad (40)\end{aligned}$$

By Equations (39) and (40), we obtain Equation (38).

Next, we assume that the Grundy number of chocolate bars $CB(F, x_1, x_2, \ldots, x_s, y)$ is

$$\mathcal{G}(\{x_1, x_2, \ldots, x_s, y\}) = x_1 \oplus x_2 \oplus \cdots \oplus x_s \oplus y. \qquad (41)$$

Let $x_2, \ldots, x_s \in \mathbb{Z}_{\geq 0}$ and $g(x_1) = F_{x_2,\ldots,x_s}(x_1) = F(x_1, x_2, \ldots, x_s)$ for $x_1 \in \mathbb{Z}_{\geq 0}$. It is sufficient to prove that g has the NS property. We assume that $j \in \mathbb{N}$. It is sufficient to prove

$$\left\lfloor \frac{g(a)}{2^{j-1}} \right\rfloor = \left\lfloor \frac{g(a+1)}{2^{j-1}} \right\rfloor$$

for $a \in \mathbb{Z}_{\geq 0}$ such that

$$\left\lfloor \frac{a}{2^j} \right\rfloor = \left\lfloor \frac{a+1}{2^j} \right\rfloor. \qquad (42)$$

We prove this by contradiction; hence, we assume that

$$\left\lfloor \frac{g(a)}{2^{j-1}} \right\rfloor < \left\lfloor \frac{g(a+1)}{2^{j-1}} \right\rfloor \qquad (43)$$

for $a \in \mathbb{Z}_{\geq 0}$, satisfying Equation (42). Here, we assume that $a \in \mathbb{Z}_{\geq 0}$ is the smallest integer that satisfies Equation (42) and Inequality (43). From Inequality (43) and (b) in Lemma 8, there exist $i, c \in \mathbb{Z}_{\geq 0}$ and $t \in R$ such that $i \geq j - 1$,

$$0 \leq t < 2^i \qquad (44)$$

and

$$g(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i \leq g(a+1). \qquad (45)$$

As $i + 1 \geq j$, according to Equation (42),

$$\left\lfloor \frac{a}{2^{i+1}} \right\rfloor = \left\lfloor \frac{a+1}{2^{i+1}} \right\rfloor.$$

Hence, from (a) in Lemma 8, for $d \in \mathbb{Z}_{\geq 0}$,

$$d \times 2^{i+1} \leq a < a + 1 < (d+1)2^{i+1}. \qquad (46)$$

From Inequality (46), if $d \times 2^{i+1} + 2^i \leq a + 1 < (d+1) \times 2^{i+1}$, then we obtain

$$d \times 2^{i+1} \leq a < a + 1 = d \times 2^{i+1} + 2^i + e < (d+1)2^{i+1} \qquad (47)$$

for $e \in \mathbb{Z}_{\geq 0}$ such that $0 \leq e < 2^i$. From Inequality (46), if $d \times 2^{i+1} < a + 1 < d \times 2^{i+1} + 2^i$, then we obtain

$$d \times 2^{i+1} \leq a < a + 1 = d \times 2^{i+1} + e < (d+1)2^{i+1} \tag{48}$$

for $e \in \mathbb{Z}_{\geq 0}$ such that $0 < e < 2^i$.

`Case 1`: If we have Inequality (47), then by Inequality (44)

$$\begin{aligned}(c \times 2^{i+1} + 2^i) \oplus (a+1) =&(c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + 2^i + e)\\ =&(c \oplus d)2^{i+1} + e\\ <&(c \oplus d)2^{i+1} + 2^i + (t \oplus e)\\ =&(c \times 2^{i+1} + t) \oplus (d \times 2^{i+1} + 2^i + e)\\ =&(c \times 2^{i+1} + t) \oplus (a+1).\end{aligned} \tag{49}$$

Let $g'(z) = \min(g(z), c \times 2^{i+1} + t)$. Because $g(z)$ increases monotonically, by Inequality (45) for $z \leq a$,

$$g(z) \leq g(a) = c \times 2^{i+1} + t.$$

Hence, for $z \leq a$

$$g'(z) = \min(g(z), c \times 2^{i+1} + t) = g(z). \tag{50}$$

By Inequality (45) and the definition of g',

$$g'(a+1) = \min(g(a+1), c \times 2^{i+1} + t) = c \times 2^{i+1} + t. \tag{51}$$

As $a \in \mathbb{Z}_{\geq 0}$ is the smallest integer satisfying Equation (42) and Inequality (43), $g'(z)$ satisfies the NS property for $z \leq a$. From Equation (51),

$$g'(a+1) = g'(a).$$

Thus, $g'(z)$ satisfies the NS property of $z \leq a+1$. We use (a) in Lemma 3 for g'. Then, we obtain

$$(a+1) \oplus (c \times 2^{i+1} + 2^i) = w \oplus \min(g'(w), c \times (2^{i+1} + t) = g'(w) \oplus w \tag{52}$$

for some $w < a + 1$ or

$$(a+1) \oplus (c \times 2^{i+1} + 2^i) = (a+1) \oplus v \tag{53}$$

for some $v < 2^{i+1} + t < 2^{i+1} + 2^i$. As Equation (53) is impossible, we obtain Equation (52).

By Inequality (45),

$$c \times 2^{i+1} + 2^i \leq g(a+1) = F(a+1, x_2, x_3, \ldots, x_s).$$

Hence, $(a+1, x_2, \dots, x_s, c \times 2^{i+1} + 2^i)$ is a position of chocolate bar $CB(F, x_1, x_2, \dots, x_s, y)$. Therefore, from Equation (41) we obtain

$$\mathcal{G}(a+1, x_2, \dots, x_s, c \times 2^{i+1} + 2^i) = (a+1) \oplus x_2 \oplus \cdots \oplus x_s \oplus (c \times 2^{i+1} + 2^i). \quad (54)$$

From Relation (52),

$$(a+1) \oplus x_2 \oplus \cdots \oplus x_s \oplus (c \times 2^{i+1} + 2^i) \in \{w \oplus x_2 \oplus \cdots \oplus x_s \oplus g'(w) : 0 \leq w \leq a\}. \quad (55)$$

Subsequently, according to Equations (41) and (50) and the definition of g,

$$\begin{aligned}
&\{w \oplus x_2 \oplus \cdots \oplus x_s \oplus g'(w) : 0 \leq w \leq a\} \\
&\quad = \{w \oplus x_2 \oplus \cdots \oplus x_s \oplus g(w) : 0 \leq w \leq a\} \\
&\quad = \{w \oplus x_2 \oplus \cdots \oplus x_s \oplus F(w, x_2, \dots, x_s) : 0 \leq w \leq a\} \\
&\quad = \{\mathcal{G}(w, x_2, \dots, x_s, F(w, x_2, \dots, x_s)) : 0 \leq w \leq a\}. \qquad (56)
\end{aligned}$$

From Inequality (45), for $w \leq a$, we have

$$F(w, x_2, \dots, x_s) \leq F(a, x_2, \dots, x_s) = g(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i.$$

Hence,

$$\begin{aligned}
&\{(w, x_2, \dots, x_s, F(w, x_2, \dots, x_s)) : 0 \leq w \leq a\} \\
&\quad = \{(w, x_2, \dots, x_s, \min(c \times 2^{i+1} + 2^i, F(w, x_2, \dots, x_s))) : 0 \leq w \leq a\} \\
&\quad = \mathit{move}(a+1, x_2, \dots, x_s, c \times 2^{i+1} + 2^i). \qquad (57)
\end{aligned}$$

From Equations (54) and (56), there exists w such that $0 \leq w \leq a$ and

$$\mathcal{G}(w, x_2, \dots, x_s, F(w, x_2, \dots, x_s)) = (a+1) \oplus x_2 \oplus \cdots \oplus x_s \oplus (c \times 2^{i+1} + 2^i). \quad (58)$$

Equations (54) and Relation (57) contradict the definition of the Grundy number.

`Case 2`: If we have Inequality (48), then, as $0 < e < 2^i$ and $0 \leq t < 2^i$,

$$\begin{aligned}
a \oplus (c \times 2^{i+1} + t) =& (d \times 2^{i+1} + e - 1) \oplus (c \times 2^{i+1} + t) \\
&= (d \times 2^{i+1} + 2^i + t \oplus (e-1)) \oplus (c \times 2^{i+1} + 2^i). \qquad (59)
\end{aligned}$$

Therefore, from Equations (41) and (59), we obtain

$$\mathcal{G}(a, x_2, \dots, x_s, c \times 2^{i+1} + t) = \mathcal{G}(d \times 2^{i+1} + 2^i + t \oplus (e-1), x_2, \dots, x_s, c \times 2^{i+1} + 2^i). \quad (60)$$

From Inequalities (45) and (48), we obtain

$$\begin{aligned}
c \times 2^{i+1} + 2^i \leq& g(a+1) \\
=& F(a+1, x_2, \dots, x_s) \\
\leq& F(a+1+t \oplus (e-1), x_2, \dots, x_s) \\
\leq& F(d \times 2^{i+1} + 2^i + t \oplus (e-1), x_2, \dots, x_s);
\end{aligned}$$

hence, $\{d \times 2^{i+1} + 2^i + t \oplus (e-1), x_2, \ldots, x_s, c \times 2^{i+1} + 2^i\}$ is the position of chocolate bar $CB(F, x_1 \ldots, x_s)$. By Inequality (45),

$$c \times 2^{i+1} + t = g(a) = F(a, x_2, \ldots, x_s). \tag{61}$$

By Inequality (48),

$$a < d \times 2^{i+1} + 2^i + t \oplus (e-1). \tag{62}$$

From Equation (61) and Inequality (62), we obtain

$$(a, x_2, \ldots, x_s, c \times 2^{i+1} + t) \in move(d \times 2^{i+1} + 2^i + t \oplus (e-1), x_2, \ldots, x_s, c \times 2^{i+1} + 2^i),$$

and this relation and Equation (60) lead to a contradiction. □

Example 5. Let $F(x, y) = \min(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{4} \rfloor)$. For fixed x_1 and y_1, by Lemmas 2 and 7, $F(x_1, y) = \min(\lfloor \frac{x_1}{2} \rfloor, \lfloor \frac{y}{4} \rfloor)$ and $F(x, y_1) = \min(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y_1}{4} \rfloor)$ satisfy the NS property, and hence by Theorem 7, for chocolate bars in Figure 26 and Figure 26, we have $\mathcal{G}(5, 9, 2) = 5 \oplus 9 \oplus 2 = 14$ and $\mathcal{G}(3, 6, 1) = 3 \oplus 6 \oplus 1 = 4$.

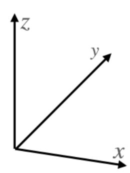

Figure 25: coordinates

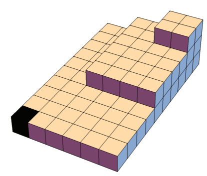

Figure 26: $(5, 9, 2)$

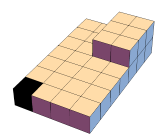

Figure 27: $(3, 6, 1)$

6. Application to the Game of Nim with a Pass

In this section, we apply the results of the multidimensional chocolate bar games to the game of Nim with a pass move in Definition 19. We assume that s denotes a fixed positive integer. We define the game of Nim with a pass move.

Definition 19. There exist s piles of stones. Two players take turns to remove stones from one of the piles. The player who removes the last stone wins the game. Here, we modified the standard rules of the game of Nim to allow a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position, that is, when there is no stone left. Once either player uses a pass, it is no longer available.

In classical Nim, the introduction of a pass alters the underlying structure of the game, significantly increasing its complexity. The formula for $\mathcal{P}$-positions of the three-pile Nim with pass is still unknown.

We generalize Nim using the pass move in Definition 20.

Definition 20. Suppose that there are s stone piles. We describe the position of this game with $(x_1, x_2, \ldots, x_s, p)$, where x_i denotes the number of stones in the i-th pile for $i = 1, 2, \ldots, s$, and we define $p = 1$ when a pass is not used, and $p = 0$ when a pass has already been used. Let $F(x_1, x_2, \ldots, x_s) \in \mathbb{Z}_{\geq 0}$ be a function of $x_1, x_2, \ldots, x_s \in \mathbb{Z}_{\geq 0}$ such that $F(x_1, x_2, \ldots, x_s) \leq 1$. For each position $(x_1, x_2, \ldots, x_s, p)$, the next player can use a pass move if and only if $F(x_1, x_2, \ldots, x_s) = 1$ and $p = 1$.

Example 6. If we define

$$F(x_1, x_2, \ldots, x_s) = \begin{cases} 0 & \text{if } x_1 + x_2 + \cdots + x_s = 0, \\ 1 & \text{if } x_1 + x_2 + \cdots + x_s > 0, \end{cases}$$

then the game in Definition 20 is the same as that in Definition 19. Therefore, the game in Definition 20 is a generalization of the game in Definition 19.

Definition 21. For $m \in \mathbb{Z}_{\geq 0}$, let

$$I_{[m,\infty)}(x) = \begin{cases} 0 & \text{if } x < m, \\ 1 & \text{if } x \geq m. \end{cases}$$

Lemma 9. *Suppose that $f(x)$ is a monotonically increasing function of $\mathbb{Z}_{\geq 0}$ into $\mathbb{Z}_{\geq 0}$ such that $f(x) \leq 1$ for any $x \in \mathbb{Z}_{\geq 0}$ and $f(x) = 1$ for some $x \in \mathbb{Z}_{\geq 0}$. Then,*

$$f(x) = I_{[m,\infty)}$$

for an even number $m \in \mathbb{Z}_{\geq 0}$ if and only if f has the NS property.

Proof. Let f be a monotonically increasing function of $\mathbb{Z}_{\geq 0}$ into $\mathbb{Z}_{\geq 0}$ such that f has the NS property, $f(x) \leq 1$ for any $x \in \mathbb{Z}_{\geq 0}$ and $f(x) = 1$ for some $x \in \mathbb{Z}_{\geq 0}$. Then, there exists a number $m \in \mathbb{Z}_{\geq 0}$ such that $f(x) = I_{[m,\infty)}$. If m is odd, then

$$\left\lfloor \frac{m-1}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor.$$

Therefore, by Definition 11, $f(m-1) = f(m)$. This contradicts $f(x) = I_{[m,\infty)}$. Therefore, m is even.

Suppose that $f = I_{[m,\infty)}$ for an even number m. Then, it is easy to observe that f satisfies the NS property. □

Corollary 1. *Let F be a monotonically increasing function such that $0 \leq F(x_1, x_2, \ldots, x_s) \leq 1$. Then,*

$$F_{x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_s}(x) = I_{[m_{x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_s},\infty)}(x)$$

for even numbers $m_{x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_s} \in \mathbb{Z}_{\geq 0}$ if and only if the Grundy number of chocolate bar $CB(F, x_1, x_2, \ldots, x_s, y)$ is

$$\mathcal{G}(x_1, x_2, \ldots, x_s, y) = x_1 \oplus x_2 \oplus \cdots \oplus x_s \oplus y.$$

Proof. This theorem is derived directly from Lemma 9 and Theorem 7. □

Lemma 10. *Let* $m_i \in \mathbb{Z}_{\geq 0}$ *for* $i = 0, 1, \ldots, s$*, and let*

$$F(x_1, x_2, \ldots, x_s) = \begin{cases} 1 & \text{if } x_i \geq m_i \text{ for some } i, \\ 0 & \text{if } x_i < m_i \text{ for all } i. \end{cases}$$

Then, $F_{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s}(x)$ *has the NS property for* $i = 1, 2, \ldots, s$ *if and only if* m_i *is even for* $i = 1, 2, \ldots, s$.

Proof. For any $i \in \mathbb{N}$ such that $1 \leq i \leq s$,

$$F_{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s}(x) = \begin{cases} 1 & \text{if } x_j \geq m_j \text{ for some } j \in \mathbb{N} \text{ such that} \\ & \ 1 \leq j \leq s \text{ and } j \neq i, \\ I_{[m_i, \infty)} & \text{if } x_j < m_j \text{ for all } j \in \mathbb{N} \text{ such that} \\ & \ 1 \leq j \leq s \text{ and } j \neq i. \end{cases}$$

Hence, by Lemma 9, we complete the proof of this lemma. □

Corollary 2. *Suppose there are* s *stone piles. Let* m_i *be a nonnegative number for* $i = 1, 2, \ldots, s$*. The players take turns removing the stones from one of the piles. The player who removes the last stone wins. We denote by* x_i *the number of stones in the* i*-th pile, and we define* $p = 1$ *when a pass is available and* $p = 0$ *when a pass has been used. We assume that a pass move is allowed in this game only when* $x_i \geq m_i$ *for some* i *and* $p = 1$*. Thus, the Grundy number of this game is*

$$\mathcal{G}(x_1, x_2, \ldots, x_s) = x_1 \oplus x_2 \oplus \cdots \oplus x_s \oplus p$$

if and only if m_i *is even for* $i = 1, 2, \ldots, s$.

Proof. This corollary is derived from Corollary 1, Definition 21, and Lemma 10. □

Corollary 2 explains why the game of Nim with a pass of Definition 19 is difficult to analyze. This is because the formula for the Grundy number is expressed as the Nim-sum of the coordinates and p only when m_i is even for $i = 1, 2, \ldots, s$. Note that $m_i = 1$ for $i = 1, 2, \ldots, s$ in the game of Nim with a pass of Definition 19.

We can calculate the Grundy numbers of this game if we change the conditions under which we use the pass move. However, this knowledge does not help us find formulas for the $\mathcal{P}$-position or Grundy number in the original three-pile Nim with a pass.

In this study, we investigate the necessary and sufficient conditions under which a chocolate bar may have a Grundy number expressed by the Nim-sum of the position coordinates.

Another question needs to be answered.

Question 4. *What is the necessary and sufficient condition under which the Nim-sum of the coordinates of the position is* 0 *if and only if the position is a* $\mathcal{P}$*-position?*

We have studied this question, but we have not found any answer. An easier question is Question 5.

Question 5. *Can you find a sufficient condition under which the Nim-sum of the coordinates of the position is* 0 *if and only if the position is a* $\mathcal{P}$*-position?*

We answer Question 5 with the following theorem.

Theorem 8 ([5])**.** *Let* $F(w,x) = \left\lfloor \frac{w+x}{k} \right\rfloor$ *for* $k = 4m+3$*, where* m *is a nonnegative integer. Then, the chocolate bar* $CB(F,0,w,x,y,0)$ *is a* $\mathcal{P}$*-position if and only if* $w \oplus x \oplus y = 0$*.*

The above theorem is applicable to three-dimensional chocolate bars, and our next aim is to generalize this theorem to the case of $(s+1)$-dimensional chocolate bars, where s is larger than 2.

6.1. Two-pile Nim with a Pass

Here, we study some examples of two-pile Nim with a pass, because we can study them as three-dimensional chocolates.

The following definition is a special case of Definition 20.

Definition 22. There are two stone piles. The players take turns removing as many stones as they prefer from one pile. The player who removes the last stone wins. We denote by x and y the numbers of stones in the piles, and let $F(x,y) \in \mathbb{Z}_{\geq 0}$ be a function of $x, y \in \mathbb{Z}_{\geq 0}$ such that $0 \leq F(x,y) \leq 1$. We assume that a pass move is allowed only in the game when $F(x,y) = 1$ and $p = 1$.

Example 7. Let t_1 and t_2 be nonnegative integers. Here we suppose that

$$F(x,y) = \begin{cases} 1 & \text{if } x \geq t_1 \text{ or } y \geq t_2, \\ 0 & \text{if } x < t_1 \text{ and } y < t_2, \end{cases}$$

where the cases of the two-pile Nim with a pass are presented in Figures 28, 29, 30, and 31. Here, the pass move is denoted by height. As observed, games of Nim with a pass move are represented as 3D chocolate games.

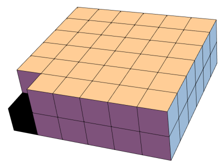

Figure 28: $t_1 = t_2 = 1$

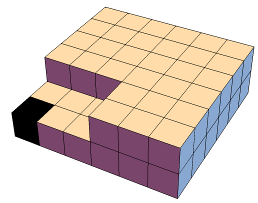

Figure 29: $t_1 = 3, t_2 = 2$

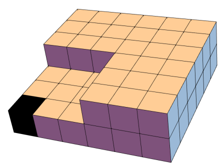

Figure 30: $t_1 = 3, t_2 = 3$

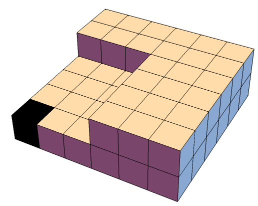

Figure 31: $t_1 = 3, t_2 = 4$

For additional research on combinatorial games with passes, see [2], [4] and [6].

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