WYTHOFF'S GAME WITH A PASS

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Abstract

This paper describes Wythoff's game with a pass, which is a variant of the classical Wythoff's game. The classical form is played with two piles of stones, from which two players take turns to remove stones from one or both piles. When removing stones from both piles, an equal number must be removed from each pile. The player who removes the last stone or stones wins. In Wythoff's game with a pass, we modify the standard rules to allow for a one-time pass, that is, a pass move that may be used at most once in a game but not from the terminal position. Once either player uses a pass, it is no longer available We denote the position of the game by (x, y, p), where x and y are the numbers of stones in the two piles, $p = 1$ if a pass is available, and $p = 0$ otherwise. The authors prove that for $(x, y, 1)$ with $x \geq 9$ or $y \geq 9$, $(x, y, 1)$ is a P-position (the previous player's winning position) if and only if the Grundy number of $(x, y, 0)$ is 1. They also prove, using the result of U. Blass and A.S. Fraenkel, that the Euclidean distance between each previous player's winning position in Wythoff's game with a pass and a nearby previous player's winning position in Wythoff's game without a pass is within $\sqrt{20}$.

1. Introduction

Let $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ be the sets of non-negative integers and natural numbers, respectively. An interesting but challenging question in combinatorial game theory has

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been determining what happens when standard game rules are modified to allow a *one-time pass*. This pass move may be used at most once in the game and not from the terminal position. Once either player has used a pass, it is no longer available for use. In the case of classical Nim, the introduction of the pass alters the mathematical structure of the game, considerably increasing its complexity, and finding the formula that describes the set of previous players' positions remains an important open question that has defied traditional approaches to solving it.

The late mathematician David Gale offered a monetary prize to the first person who developed a solution for the three-pile classical Nim with a pass. In [9] (p. 370), Morrison, Friedman, and Landsberg conjectured that "solvable combinatorial games are structurally unstable to perturbations, while generic, complex games will be structurally stable." One way to introduce such a perturbation is to allow a pass. However, the authors of the present article reported some games as counterexamples to this conjecture in [5], [7], and [8]. These games are solvable because there are simple formulas for the Grundy numbers, and even when we introduce a pass move to the games, there are simple formulas for $\mathcal{P}$-positions. Based on the research in [5], [7], and [8], the authors of the present article propose the following view on the combinatorial game with a pass.

Some games have specific mathematical structures that prevent the perturbation caused by the pass from spreading to other positions, and these games have formulas for $\mathcal{P}$-positions, even if a pass is introduced. However, the mathematical structures of some games permit the perturbation caused by the pass to spread all over the positions.

Here, we present research on Wythoff's game with a pass. Wythoff's game with a pass presents a perfect example of specific mathematical structures that prevent the perturbation caused by the pass from spreading to other positions.

For other research on combinatorial games with a pass, see [3], [4], and [6]. In [3] and [6], Chan, Low, Locke, and Wong described the set of previous players' positions of Nim with a pass when the number of stones in each pile is at most four. This study shows that the impact of perturbation is small when the number of stones in each pile is small. In [4], it was proven that the arithmetic periodicity of the $\mathcal{G}$-sequence can occur when we add a single pass move to precisely one pile in finite octal games, although finite octal games are not arithmetic periodic. Therefore, in this case, regularity, not perturbation, occurs by adding a single pass to the finite octal games.

For completeness, we briefly review some of the necessary concepts in combinatorial game theory by referring to [1] and [10].

Definition 1. Let x and y be non-negative integers. We represent them in base 2, so that $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$. We define the nim-sum $x \oplus y$ by

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i,$$

where $w_i = x_i + y_i \pmod 2$.

Wythoff's game is an impartial game without drawings; only two outcome classes are possible.

Definition 2. A position is referred to as a P-position if it is the winning position for the previous player (the player who has just moved), as long as the player plays correctly at each stage. A position is referred to as an N-position if it is the winning position for the next player, as long as they play correctly at each stage.

Definition 3. The *disjunctive sum* of the two games, denoted by $\mathbf{G}+\mathbf{H}$, is a super game in which a player may move either in $\mathbf{G}$ or $\mathbf{H}$ but not in both.

Definition 4. For any position $\mathbf{p}$ in game $\mathbf{G}$, a set of positions can be reached by a single move in $\mathbf{G}$, which we denote as $move(\mathbf{p})$.

Definition 5. The *minimum excluded value* (*mex*) of a set S of nonnegative integers is the least nonnegative integer that is not in S.

Definition 6. Let $\mathbf{p}$ be a position in the impartial game. The associated *Grundy number* is denoted by $G(\mathbf{p})$ and is recursively defined by $G(\mathbf{p}) = mex(\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

The next result demonstrates the usefulness of the Sprague–Grundy theory for impartial games.

Theorem 1 ([1])**.** *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be impartial rulesets, and* $G_{\mathbf{G}}$ *and* $G_{\mathbf{H}}$ *be the Grundy numbers of game* $\mathbf{g}$ *played under the rules of* $\mathbf{G}$ *and game* $\mathbf{h}$ *played under those of* $\mathbf{H}$*. Then, we obtain the following:*
(*i*) *for any position* $\mathbf{g}$ *in* $\mathbf{G}$*, we have that* $G_{\mathbf{G}}(\mathbf{g}) = 0$ *if and only if* $\mathbf{g}$ *is the* $\mathcal{P}$*-position;*
(*ii*) *the Grundy number of positions* $\{\mathbf{g}, \mathbf{h}\}$ *in game* $\mathbf{G}+\mathbf{H}$ *is* $G_{\mathbf{G}}(\mathbf{g}) \oplus G_{\mathbf{H}}(\mathbf{h})$.

Using Theorem 1, we can determine the $\mathcal{P}$-position by calculating the Grundy numbers and the $\mathcal{P}$-position of the sum of the two games by calculating the Grundy numbers of the two games.

2. Wythoff's Game

In this section, we review some of the theorems of Wyhoff's game for later use. For the details of Wythoff's game, see [11].

Definition 7. Wythoff's game is played with two piles of stones. Two players take turns removing stones from one or both piles. When removing stones from both piles, the number of stones removed from each pile should be equal. The player who removes the last stone or stones wins. An equivalent description of the game is that a single chess queen is placed somewhere on a large grid of squares, and each player can move the queen towards the upper-left corner of the grid, either vertically, horizontally, or diagonally, for any number of steps. The winner is the player who moves the queen to the upper-left corner.

Figure 1 shows the grid of squares, and we denote by (x, y) the number of stones in the first and second piles or the position of the queen, where the horizontal and vertical coordinates are denoted by x and y. Figure 2 shows the moves that the queen can make in Wythoff's game.

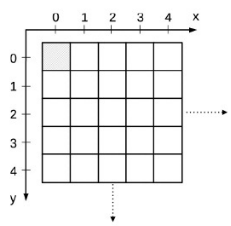

Figure 1

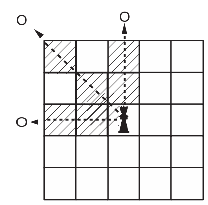

Figure 2

Theorem 2 ([11]). *The set of $\mathcal{P}$-positions of the game in Definition 7 is*

$$\{(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n) : n \in \mathbb{Z}_{\geq 0}\} \cup \{(\lfloor n\phi \rfloor + n, \lfloor n\phi \rfloor) : n \in \mathbb{Z}_{\geq 0}\},$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

Theorem 2 is a well-known fact in Wythoff's game.

Theorem 3 ([2]). *Let* $\{(a_n, b_n) : n \in \mathbb{Z}_{\geq 0}\} = \{(x, y) : \mathcal{G}(x, y) = 1\}$. *Here, we assume that* a_n *is increasing. Then, we obtain*

$$\mid b_n - (\lfloor n\phi \rfloor + n) \mid \leq 4$$

and

$$\lfloor n\phi \rfloor - 1 \leq a_n \leq \lfloor n\phi \rfloor + 2.$$

Theorem 3 is Corollary 5.14 of [2].

Corollary 1. *For any position* (x, y) *with* $\mathcal{G}(x, y) = 1$, *there exists a position* (v, w) *such that* $\mathcal{G}(v, w) = 0$ *and* $\sqrt{(x - v)^2 + (y - w)^2} \leq \sqrt{20}$.

Proof. This follows directly from Theorems 2 and 3. □

3. Wythoff's Game with a Pass and the Sum of Wythoff's Game and a Pile of One Stone

In this section, we define a new variant of Wythoff's game and compare it to the sum of Wythoff's game and a pile of one stone.

Definition 8. *Wythoff's game with a pass* is played like the ordinary Wythoff's game, with the option of a single pass that can be used by exactly one player. Once a pass is used, it cannot be used again. The pass can be used at any time up to the penultimate move, but it cannot be used at the end of the game. The player who cannot make a move loses. We denote by $\mathcal{G}_1$ the Grundy number of this game.

Here, we introduce the sum of the traditional Wythoff's game and a pile of one stone. We need this game to study $\mathcal{P}$-positions of Wythoff's Game with a Pass.

Definition 9. Applying Definition 3, we define the sum of the classical Wythoff's game without a pass and the game of a pile of one stone. We denote by $\mathcal{G}_2$ the Grundy number of this game.

We denote the position of the game in Definition 8 and the game in Definition 9 by three coordinates $\{x, y, p\}$. The coordinates x, y define the number of stones in the first and second piles, or, if we use a queen in the game, the position of the queen on the chessboard. For the game in Definition 8, the additional parameter p denotes whether the pass is still available ($p = 1$) or has already been used ($p = 0$). For the game in Definition 9, the parameter $p = 1$ if there is a stone in the third pile, and $p = 0$ if there is no stone in the third pile. Note that when $p = 0$, the games in Definitions 8 and 9 are the classical Wythoff's game.

Definition 10. For any $x, y \in \mathbb{Z}_{\geq 0}$ and $p = 0, 1$, let

$$M_1(x, y, p) = \{(u, y, p) : u < x \text{ and } u \in \mathbb{Z}_{\geq 0}\},$$

$$M_2(x, y, p) = \{(x, v, p) : v < y \text{ and } v \in \mathbb{Z}_{\geq 0}\},$$

$$M_3(x, y, p) = \{(x - t, y - t, p) : 1 \leq t \leq \min(x, y) \text{ and } t \in \mathbb{Z}_{\geq 0}\},$$

$$M_4(x, y, p) = \begin{cases} \{(x, y, 0)\} & (\text{ if } x + y > 0 \text{ and } p = 1), \\ \emptyset & (\text{ if } x + y = 0 \text{ or } p = 0), \end{cases}$$

and

$$M'_4(x, y, p) = \begin{cases} \{(x, y, 0)\} & (\text{ if } p = 1), \\ \emptyset & (\text{ if } p = 0). \end{cases}$$

The sets $M_1(x,y,p)$, $M_2(x,y,p)$, and $M_3(x,y,p)$ are the sets of horizontal, vertical, and diagonal moves, respectively. Set $M_4(x,y,p)$ is the set of the pass move of Wythoff's game with a pass in Definition 8, and Set $M_4'(x,y,p)$ is the set of moves in the third pile of the game in Definition 9. Note that $M_4(x,y,p)$ is empty if and only if $x+y=0$ or $p=0$, and $M_4'(x,y,p)$ is empty if and only if $p=0$.

Next, we define $move_1$ and $move_2$, which are moves of the games in Definitions 8 and 9, respectively.

Definition 11. For any $x, y \in \mathbb{Z}_{\geq 0}$ and $p = 0, 1$, let

$$move_1(x,y,p) = M_1(x,y,p) \cup M_2(x,y,p) \cup M_3(x,y,p) \cup M_4(x,y,p)$$

and

$$move_2(x,y,p) = M_1(x,y,p) \cup M_2(x,y,p) \cup M_3(x,y,p) \cup M_4'(x,y,p).$$

4. The Positions (x, y, p) such that $x, y \leq 8$

This section aims to determine $\mathcal{P}$-positions and $\mathcal{N}$-positions in the set $\{(x,y,p) : x, y \leq 8\}$ in Lemma 1. We now define three sets, A, B, and C, and study them. These sets have mathematical structures that prevent the perturbation caused by the pass from spreading to other positions. In Figures 3, 4, 5, 6, 7, 8, 9, the Grundy numbers of each point are printed, but these Grundy numbers have nothing to do with the argument in this section.

Definition 12. Let

$$A = \{(0,0,0), (1,2,0), (2,1,0), (3,5,0), (4,7,0), (5,3,0), (7,4,0)\},$$

$$B = \{(0,0,1), (1,3,1), (3,1,1), (2,5,1), (5,2,1), (4,8,1), (8,4,1), (6,7,1), (7,6,1)\},$$

and

$$C = \{(0,1,1), (1,0,1), (2,2,1), (3,6,1), (6,3,1), (4,8,1), (8,4,1), (5,7,1), (7,5,1)\}.$$

In Figures 3, 4, and 5, we have the sets of Grundy numbers $\{\mathcal{G}_1(x,y,0) : x, y \leq 8\}$, $\{\mathcal{G}_1(x,y,1) : x, y \leq 8\}$, and $\{\mathcal{G}_2(x,y,1) : x, y \leq 8\}$, respectively. Here, sets A, B, and C are printed in red.

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	2	0	1	9	10
4	4	5	3	2	7	6	9	0	1
5	5	3	4	0	6	8	10	1	2
6	6	7	8	1	9	10	3	4	5
7	7	8	6	9	0	1	4	5	3
8	8	6	7	10	1	2	5	3	4

Figure 3: Set A

	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	2	1	3	0	6	4	8	7	5
2	1	3	2	6	4	0	7	5	8
3	4	0	6	3	1	2	9	10	11
4	3	6	4	1	5	7	10	2	0
5	6	4	0	2	7	9	11	3	1
6	5	8	7	9	10	11	4	0	6
7	8	7	5	10	2	3	0	6	4
8	7	5	8	11	0	1	6	4	10

Figure 4: Set B

	0	1	2	3	4	5	6	7	8
0	1	0	3	2	5	4	7	6	9
1	0	3	1	5	4	2	6	9	7
2	3	1	0	4	2	5	9	7	6
3	2	5	4	7	3	1	0	8	11
4	5	4	2	3	6	7	8	1	0
5	4	2	5	1	7	9	11	0	3
6	7	6	9	0	8	11	2	5	4
7	6	9	7	8	1	0	5	4	2
8	9	7	6	11	0	3	4	2	5

Figure 5: Set C

	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	2	1	3	0	6	4	8	7	5
2	1	3	2	6	4	0	7	5	8
3	4	0	6	3	1	2	9	10	11
4	3	6	4	1	5	7	10	2	0
5	6	4	0	2	7	9	11	3	1
6	5	8	7	9	10	11	4	0	6
7	8	7	5	10	2	3	0	6	4
8	7	5	8	11	0	1	6	4	10

Figure 6: From P not P

	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	2	1	3	0	6	4	8	7	5
2	1	3	2	6	4	0	7	5	8
3	4	0	6	3	1	2	9	10	11
4	3	6	4	1	5	7	10	2	0
5	6	4	0	2	7	9	11	3	1
6	5	8	7	9	10	11	4	0	6
7	8	7	5	10	2	3	0	6	4
8	7	5	8	11	0	1	6	4	10

Figure 7: Horizontal

	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	2	1	3	0	6	4	8	7	5
2	1	3	2	6	4	0	7	5	8
3	4	0	6	3	1	2	9	10	11
4	3	6	4	1	5	7	10	2	0
5	6	4	0	2	7	9	11	3	1
6	5	8	7	9	10	11	4	0	6
7	8	7	5	10	2	3	0	6	4
8	7	5	8	11	0	1	6	4	10

Figure 8: Vertical

	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	2	1	3	0	6	4	8	7	5
2	1	3	2	6	4	0	7	5	8
3	4	0	6	3	1	2	9	10	11
4	3	6	4	1	5	7	10	2	0
5	6	4	0	2	7	9	11	3	1
6	5	8	7	9	10	11	4	0	6
7	8	7	5	10	2	3	0	6	4
8	7	5	8	11	0	1	6	4	10

Figure 9: Daiagonal

Lemma 1. *(i) The set A in Definition 12 is the set of $\mathcal{P}$-positions $(x, y, 0)$ of the game in Definition 8 such that $x, y \leq 8$ and the pass is not available.*
(ii) The set B in Definition 12 is the set of $\mathcal{P}$-positions $(x, y, 1)$ of the game in Definition 8 such that $x, y \leq 8$ and the pass is available.
(iii) The set C in Definition 12 is the set of $\mathcal{P}$-positions $(x, y, 1)$ of the game in Definition 9 such that $x, y \leq 8$ and the third coordinate is 1.

Proof. (i) Since the pass is not available, by using Theorem 2 for $x, y \leq 8$ we obtain Set A.

(ii) Let $U = \{(x, y, 1) : x, y \leq 8\}$. Since we need to prove that the set $\mathcal{P}$-positions of the game in Definitions 8 in U is B when a pass is available, we need to prove that

$$move_1(x, y, 1) \cap (A \cup B) = \emptyset \tag{1}$$

for any $(x, y, 1) \in B$ and

$$move_1(x, y, 1) \cap (A \cup B) \neq \emptyset \tag{2}$$

for any $(x, y, 1) \in U - B$.

First, we prove Relation (1). Suppose that we start with the position $(7, 6, 1)$. Then, the horizontal, vertical, and diagonal moves from this position are described in Figure 6, and it is easy to see that $M_1(7, 6, 1) \cap B = \emptyset$, $M_2(7, 6, 1) \cap B = \emptyset$, and $M_3(7, 6, 1) \cap B = \emptyset$. Since $M_4(7, 6, 1) = \{(7, 6, 0)\}$ and $(7, 6, 0) \notin A$, we obtain $M_4(7, 6, 1) \cap A = \emptyset$. Therefore, $move_1(7, 6, 1) \cap (A \cup B) = \emptyset$. Similarly, for any $(x, y, 1) \in B$, it is easy to show that $M_1(x, y, 1) \cap B = \emptyset$, $M_2(x, y, 1) \cap B = \emptyset$, and $M_3(x, y, 1) \cap B = \emptyset$. By comparing Figure 3 and Figure 4, we obtain $M_4(x, y, 1) \cap A = \emptyset$. Therefore, we obtain Relation (1).

Next, we prove Relation (2). Let $(x, y, 1) \in U - B$. For $(8, y, 1)$ with $y = 0, 1, 2, 3$ and $y = 5, 6, 7, 8$, it is clear that $M_1(x, y, 1) \cap B \neq \emptyset$. In this way, for all blue positions $(x, y, 1)$ in Figure 7, we obtain $M_1(x, y, 1) \cap B \neq \emptyset$. Similarly, for all blue positions $(x, y, 1)$ in Figure 8, we obtain $M_2(x, y, 1) \cap B \neq \emptyset$ and for all blue positions $(x, y, 1)$ in Figure 9, we obtain $M_3(x, y, 1) \cap B \neq \emptyset$. The positions in $U - B$ that do not belong to the set of blue positions in Figures 7, 8, and 9 are $(1, 2, 1)$ and $(2, 1, 1)$. Since $(1, 2, 0)$ and $(2, 1, 0)$ belong to the set A in Figure 3, $M_4(1, 2, 1) \cap A \neq \emptyset$ and $M_4(2, 1, 1) \cap A \neq \emptyset$. Therefore, we obtain Relation (2).

(iii) By a method that is very similar to the one used in (ii), we can prove (iii). Therefore, the details are omitted. □

5. The Set of $\mathcal{P}$-positions of Wythoff's Game with a Pass

In this section, we determine the set of $\mathcal{P}$-positions of Wythoff's game with a pass. We use the similarity between the set of $\mathcal{P}$-positions of Wythoff's game with a pass and the set of $\mathcal{P}$-positions of the game in Definition 9.

Let $P_0 = \{(x, y, 0) : \mathcal{G}_1(x, y, 0) = 0\}$, $P_1 = \{(x, y, 1) : \mathcal{G}_1(x, y, 1) = 0\}$, and $P_2 = \{(x, y, 1) : \mathcal{G}_2(x, y, 1) = 0\}$.

Lemma 2. *Let $x, y \in \mathbb{Z}_{\geq 0}$ such that $x \geq 9$ or $y \geq 9$. Then, we obtain the following:*
(i) if $y \leq 8$, then $M_1(x, y, 1) \cap B \neq \emptyset$, $M_1(x, y, 1) \cap C \neq \emptyset$, and $M_2(x, y, 1) \cap C = M_2(x, y, 1) \cap B = \emptyset$;
(ii) if $x \leq 8$, then $M_2(x, y, 1) \cap B \neq \emptyset$, $M_2(x, y, 1) \cap C \neq \emptyset$, and $M_1(x, y, 1) \cap C = M_1(x, y, 1) \cap B = \emptyset$;

(iii) if $x \le y + 4$ *and* $y \le x + 4$*, then* $M_3(x, y, 1) \cap B \neq \emptyset$ *and* $M_3(x, y, 1) \cap C \neq \emptyset$*;*
(iv) if $x \ge y + 5$ *or* $y \ge x + 5$*, then* $M_3(x, y, 1) \cap B = M_3(x, y, 1) \cap C = \emptyset$*.*

Proof. (i) Suppose that $x \ge 9$ and $y \le 8$. Then, by Definition 12, there exist $u, u' \in \mathbb{Z}_{\ge 0}$ such that $1 \le u, u' \le 8$, $(u, y, 1) \in B$, and $(u', y, 1) \in C$. Then, we obtain $(u, y, 1) \in M_1(x, y, 1) \cap B$ and $(u', y, 1) \in M_1(x, y, 1) \cap C$. Since $x \ge 9$, $M_2(x, y, 1) \subset \{(x, v, 1) : v \in \mathbb{Z}_{\ge 0}\} \subset (B \cup C)^c$, where $(B \cup C)^c$ is the complement of the set $B \cup C$. Hence, $M_2(x, y, 1) \cap C = M_2(x, y, 1) \cap B = \emptyset$.
(ii) Suppose that $y \ge 9$ and $x \le 8$. Then, (ii) follows directly from (i), because this game is symmetrical with respect to the first and second coordinates.
(iii) Suppose that $x \le y + 4$ and $y \le x + 4$. By Definition 12, for any $a \in \mathbb{Z}_{\ge 0}$ such that $-4 \le a \le 4$, there exist $u, u', v, v' \in \mathbb{Z}_{\ge 0}$ such that $u = v + a$, $u' = v' + a$, $(u, v, 1) \in B$, and $(u', v', 1) \in C$. Then, $(u, v, 1) \in B \cap M_3(x, y, 1)$ and $(u', v', 1) \in C \cap M_3(x, y, 1)$. Therefore, $M_3(x, y, 1) \cap B \neq \emptyset$ and $M_3(x, y, 1) \cap C \neq \emptyset$.
(iv) We have two cases.
`Case 1`: Suppose that $x \ge y + 5$. There is no $u, v \in \mathbb{Z}_{\ge 0}$ such that $u \ge v + 5$ and $(u, v, 1) \in B \cup C$. Hence, $M_3(x, y, 1) \cap B = M_3(x, y, 1) \cap C = \emptyset$.
`Case 2`: Suppose that $y \ge x + 5$. Since this game is symmetrical with respect to the first and the second coordinates, $M_3(x, y, 1) \cap B = M_3(x, y, 1) \cap C = \emptyset$. □

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	2	1	4	3	6	5	8	7						
1	2	1	3	0	6	4	8	7	5						
2	1	3	2	6	4	0	7	5	8						
3	4	0	6	3	1	2	9	10	11						
4	3	6	4	1	5	7	10	2	0						
5	6	4	0	2	7	9	11	3	1						
6	5	8	7	9	10	11	4	0	6						
7	8	7	5	10	2	3	0	6	4						
8	7	5	8	11	0	1	6	4	10						
9															
10															
11															
12															
13															
14															

Figure 10: Set B and other positions

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	0	3	2	5	4	7	6	9						
1	0	3	1	5	4	2	6	9	7						
2	3	1	0	4	2	5	9	7	6						
3	2	5	4	7	3	1	0	8	11						
4	5	4	2	3	6	7	8	1	0						
5	4	2	5	1	7	9	11	0	3						
6	7	6	9	0	8	11	2	5	4						
7	6	9	7	8	1	0	5	4	2						
8	9	7	6	11	0	3	4	2	5						
9															
10															
11															
12															
13															
14															

Figure 11: Set C and other positions

Lemma 3. *For* $x, y \in \mathbb{Z}_{\ge 0}$ *such that* $x \ge 9$ *or* $y \ge 9$*, we obtain the following:*
(i) $M_1(x, y, 1) \cap B \neq \emptyset$ *if and only if* $M_1(x, y, 1) \cap C \neq \emptyset$*;*
(ii) $M_2(x, y, 1) \cap B \neq \emptyset$ *if and only if* $M_2(x, y, 1) \cap C \neq \emptyset$*;*
(iii) $M_3(x, y, 1) \cap B \neq \emptyset$ *if and only if* $M_3(x, y, 1) \cap C \neq \emptyset$*.*

Proof. By Lemma 2, we obtain (i), (i), and (iii). □

Theorem 4. *For $x, y \in \mathbb{Z}_{\geq 0}$ such that $x \geq 9$ or $y \geq 9$, we obtain the following relation:*

$$\mathcal{G}_1(x, y, 1) = 0 \text{ if and only if } \mathcal{G}_2(x, y, 1) = 0. \quad (3)$$

Proof. Let $V_8 = \{(x, y, 1) : x, y \leq 8\}$. It is sufficient to prove that

$$P_1 - V_8 = P_2 - V_8.$$

Let $U_k = \{(x, y, 1) : x + y \leq k\}$, and by mathematical induction we prove that

$$(U_n - V_8) \cap P_1 = (U_n - V_8) \cap P_2$$

for any natural number n. Since

$$(U_{17} - V_8) \subset \{(u, v) : u \geq 9 \text{ and } v \leq 8\} \cup \{(u, v) : u \leq 8 \text{ and } v \geq 9\},$$

by (i) and (ii) of Lemma 2, any point in $U_{17} - V_8$ is an $\mathcal{N}$-position. Hence

$$(U_{17} - V_8) \cap P_1 = (U_{17} - V_8) \cap P_2 = \emptyset.$$

For some natural number k with $k \geq 18$, we suppose that

$$(U_k - V_8) \cap P_1 = (U_k - V_8) \cap P_2. \quad (4)$$

Let $x, y \in \mathbb{Z}_{\geq 0}$ such that $(x, y, 1) \in U_{k+1} - V_8$. Then, for $i = 1, 2, 3$, by Definition 10

$$M_i(x, y, 1) \subset U_k,$$

and hence we obtain

$$\begin{aligned} M_i(x, y, 1) \cap P_1 &= M_i(x, y, 1) \cap ((U_k - V_8) \cup V_8) \cap P_1 \\ &= (M_i(x, y, 1) \cap (U_k - V_8) \cap P_1) \cup (M_i(x, y, 1) \cap V_8 \cap P_1) \\ &= (M_i(x, y, 1) \cap (U_k - V_8) \cap P_1) \cup (M_i(x, y, 1) \cap B) \quad (5) \end{aligned}$$

and

$$\begin{aligned} M_i(x, y, 1) \cap P_2 &= M_i(x, y, 1) \cap ((U_k - V_8) \cup V_8) \cap P_2 \\ &= (M_i(x, y, 1) \cap (U_k - V_8) \cap P_2) \cup (M_i(x, y, 1) \cap V_8 \cap P_2) \\ &= (M_i(x, y, 1) \cap (U_k - V_8) \cap P_2) \cup (M_i(x, y, 1) \cap C). \quad (6) \end{aligned}$$

By Lemma 3 and Equations (4), (5), and (6), we obtain

$$M_i(x, y, 1) \cap P_1 \neq \emptyset \text{ if and only if } M_i(x, y, 1) \cap P_2 \neq \emptyset \quad (7)$$

for $i = 1, 2, 3$. Since $M_4(x, y, 1) = M'_4(x, y, 1) = (x, y, 0)$ for x, y such that $x \geq 9$ or $y \geq 9$, we obtain

$$M_4(x, y, 1) \cap P_0 \neq \emptyset \text{ if and only if } M'_4(x, y, 1) \cap P_0 \neq \emptyset. \quad (8)$$

By (7) and (8), for any x, y such that $(x, y, 1) \in U_{k+1} - V_8$,

$$move_1(x, y, 1) \cap (P_1 \cup P_0) \neq \emptyset \text{ if and only if } move_2(x, y, 1) \cap (P_2 \cup P_0) \neq \emptyset,$$

and hence,

$$(U_{k+1} - V_8) \cap P_1 = (U_{k+1} - V_8) \cap P_2.$$

Therefore, by mathematical induction, we obtain

$$(U_n - V_8) \cap P_1 = (U_n - V_8) \cap P_2$$

for any natural number n. □

By Theorem 4, a position $(x, y, 1)$ is a $\mathcal{P}$-position of the game in Definition 8 if and only if it is a $\mathcal{P}$-position of the game in Definition 9 when $x \geq 9$ or $y \geq 9$. The mathematical structures of Sets A, B, and C prevent the perturbation caused by the pass from spreading to other positions.

Corollary 2. *For any $x, y \in \mathbb{Z}_{\geq 0}$ such that $x \geq 9$ or $y \geq 9$,*

$$\mathcal{G}_1(x, y, 1) = 0 \textit{ if and only if } \mathcal{G}_1(x, y, 0) = 1. \tag{9}$$

Proof. By Theorem 1 and Definition 9, $\mathcal{G}_2(x, y, 1) = \mathcal{G}_2(x, y, 0) \oplus 1 = \mathcal{G}_1(x, y, 0) \oplus 1$. By Theorem 4, $\mathcal{G}_1(x, y, 1) = \mathcal{G}_2(x, y, 1)$. Hence, we obtain Relation (9). □

Theorem 5. *For any position $(x, y, 1)$ such that $\mathcal{G}_1(x, y, 1) = 0$, there exists a position $(v, w, 0)$ such that $\mathcal{G}_1(v, w, 0) = 0$ and the Euclidean distance between (x, y) and (v, w) is within $\sqrt{20}$.*

Proof. This follows directly from Corollaries 1 and 2. □

By Theorem 5, the graph of the set of $\mathcal{P}$-positions in the classical Wythoff's game and the graph of $\mathcal{P}$-positions in Wythoff's game with a pass when the pass is still available look very similar. See Figures 5.1 and 5.2.

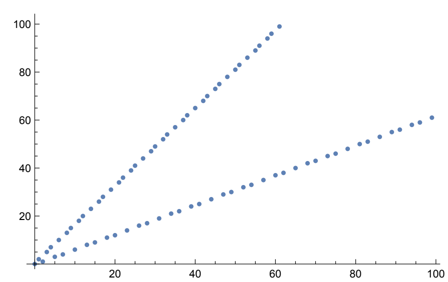

Figure 5.1. The graph of the set of $\mathcal{P}$-positions in the classical Wythoff's game.

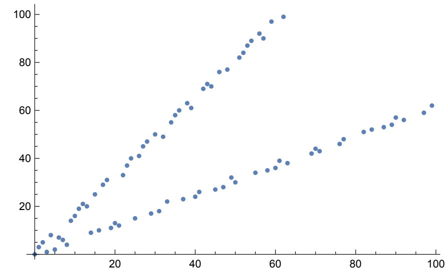

Figure 5.2. The graph of $\mathcal{P}$-positions in Wythoff's game with a pass when the pass is still available.

6. Calculation of Grundy Numbers for Wythoff's Game with a Pass by Computers

Here, we have a table of the Grundy numbers in Figures 6.1 and 6.2. We used the Combinatorial Game Suite, and in these tables, the symbol $*$ denotes the number 1.

0	*	*2	*3	*4	*5	*6	*7	*8	*9	*10	*11	*12	*13	*14	*15
*	*2	0	*4	*5	*3	*7	*8	*6	*10	*11	*9	*13	*14	*12	*16
*2	0	*	*5	*3	*4	*8	*6	*7	*11	*9	*10	*14	*12	*13	*17
*3	*4	*5	*6	*2	0	*	*9	*10	*12	*8	*7	*15	*11	*16	*18
*4	*5	*3	*2	*7	*6	*9	0	*	*8	*13	*12	*11	*16	*15	*10
*5	*3	*4	0	*6	*8	*10	*	*2	*7	*12	*14	*9	*15	*17	*13
*6	*7	*8	*	*9	*10	*3	*4	*5	*13	0	*2	*16	*17	*18	*12
*7	*8	*6	*9	0	*	*4	*5	*3	*14	*15	*13	*17	*2	*10	*19
*8	*6	*7	*10	*	*2	*5	*3	*4	*15	*16	*17	*18	0	*9	*14
*9	*10	*11	*12	*8	*7	*13	*14	*15	*16	*17	*6	*19	*5	*	0
*10	*11	*9	*8	*13	*12	0	*15	*16	*17	*14	*18	*7	*6	*2	*3
*11	*9	*10	*7	*12	*14	*2	*13	*17	*6	*18	*15	*8	*19	*20	*21
*12	*13	*14	*15	*11	*9	*16	*17	*18	*19	*7	*8	*10	*20	*21	*22
*13	*14	*12	*11	*16	*15	*17	*2	0	*5	*6	*19	*20	*9	*7	*8
*14	*12	*13	*16	*15	*17	*18	*10	*9	*	*2	*20	*21	*7	*11	*23
*15	*16	*17	*18	*10	*13	*12	*19	*14	0	*3	*21	*22	*8	*23	*20

Figure 6.1. Table of Grundy numbers for Wythoff's game without a pass.

0	*2	*	*4	*3	*6	*5	*8	*7	*10	*9	*12	*11	*14	*13	*16
*2	*	*3	0	*6	*4	*8	*7	*5	*9	*12	*10	*14	*13	*11	*15
*	*3	*2	*6	*4	0	*7	*5	*8	*12	*10	*9	*13	*11	*14	*18
*4	0	*6	*3	*	*2	*9	*10	*11	*13	*7	*5	*8	*12	*15	*17
*3	*6	*4	*	*5	*7	*10	*2	0	*14	*11	*13	*12	*15	*8	*9
*6	*4	0	*2	*7	*9	*11	*3	*	*5	*13	*15	*10	*8	*16	*14
*5	*8	*7	*9	*10	*11	*4	0	*6	*3	*	*16	*2	*18	*17	*13
*8	*7	*5	*10	*2	*3	0	*6	*4	*11	*14	*17	*	*9	*19	*20
*7	*5	*8	*11	0	*	*6	*4	*10	*16	*15	*18	*9	*2	*3	*21
*10	*9	*12	*13	*14	*5	*3	*11	*16	*7	*8	*19	*15	*6	0	*
*9	*12	*10	*7	*11	*13	*	*14	*15	*8	*16	*20	*5	*17	*18	*4
*12	*10	*9	*5	*13	*15	*16	*17	*18	*19	*20	*8	*21	*7	*22	*2
*11	*14	*13	*8	*12	*10	*2	*	*9	*15	*5	*21	*17	*19	*20	*7
*14	*13	*11	*12	*15	*8	*18	*9	*2	*6	*17	*7	*19	*20	*5	*22
*13	*11	*14	*15	*8	*16	*17	*19	*3	0	*18	*22	*20	*5	*12	*10
*16	*15	*18	*17	*9	*14	*13	*20	*21	*	*4	*2	*7	*22	*10	*11

Figure 6.2. Table of Grundy numbers for Wythoff's game with a pass.

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