Besov Spaces on $\mathbb{R}^n$

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Let $\phi \in \mathcal{S}_0 = \{f \in \mathcal{S} | \widehat{f}^{(n)}(0) = 0 \forall n\}$ be such that

$$\mathrm{supp}(\widehat{\phi}) \subseteq \{1/2 \leq |x| \leq 2\}, \quad 0 \leq \widehat{\phi} \leq 1, \quad \sum_{j \in \mathbb{Z}} \widehat{\phi}_j = 1$$

where $\widehat{\phi}_j(x) = \widehat{\phi}(2^{-j}x)$.

Then the homogeneous Besov spaces are defined by Peetre and Triebel as

$$\dot{B}_s^{p,q} = \{f \in \mathcal{S}_0' | \|f\|_{s,p,q} < \infty\}$$

where

$$\|f\|_{s,p,q} = \Big(\sum_{j \in \mathbb{Z}} 2^{sj} \|\mathcal{F}^{-1}(\widehat{\phi}_j \widehat{f})\|_{L^p}^q \Big)^{1/q}$$

Feichtinger/Gröchenig coorbit theory

Let (π, H) be a unitary irreducible integrable representation of G, and w a submultiplicative weight on G. Assume that Y is solid BF-space such that

$$Y * L^1_w(G) \subseteq Y \qquad \text{and} \qquad \|F * f\|_Y \leq C\|F\|_Y\|f\|_{1,w}$$

Let

$$u \in H^1_w = \{v \in H | V_u(v) \in L^1_w(G)\}$$

Then H^1_w is a Banach space with norm $\|v\| = \|V_u(v)\|_{1,w}$, and we can define the coorbit space

$$\mathrm{Co}_{FG} Y = \{v \in (H^1_w)^* | V_u(v) \in Y\}$$

with norm $\|v\|_{FG} = \|V_u(v)\|_Y$. These are Banach spaces.

Besov spaces as coorbits

Gröchenig has characterized the Besov spaces as coorbits of the similitude group $G = (\mathbb{R}_+ SO(n)) \ltimes \mathbb{R}^n$. With the representation

$$\pi(A,b)f(x) = |\det(A)|^{1/2} f(A^{-1}(x-b))$$

we have

$$\dot{B}^{p,q}_s = \mathrm{Co}_{FG} L^{p,q}_{s+n/2-n/q}$$

where

$$L^{p,q}_\alpha = \left\{ f \,\middle|\, \int \left(\int |f(aR,b)|^p \, db \, dR \right)^{q/p} a^{-\alpha q} \frac{da}{a^{n+1}} < \infty \right\}$$

If we choose a radial ϕ we only need the $ax+b$-group, but then the representation is not irreducible. The point is that ϕ is cyclic. Besov spaces are defined using the space $\mathcal{S}_0$, why is it the same to use H^1_w where $w(a,b) = a^{-s}$? Can we use H^∞_π instead?

Notation

- Let S be a Frechét space which is weakly dense in its conjugate dual S^* and let $\langle v', u \rangle$ denote the conjugate dual pairing of $u \in S$ and $v' \in S^*$
- Let π be a representation of a (Lie) group on S and define the voice transform $V_u(v')(x) = \langle v', \pi(x)u \rangle$
- Denote by π also the contragradient representation on S^*, i.e. $\langle \pi(x)v', u \rangle = \left\langle v', \pi(x^{-1})u \right\rangle$
- Let Y be some left- and right-invariant (quasi) Banach Function space.

A generalization (Coorbit spaces)

Assume there is a cyclic $u \in S$ such that

R1. $V_u(v) * V_u(u) = V_u(v)$ for all $v \in S$

R2. $Y * V_u(u) \subseteq Y$ and $f \mapsto f * V_u(u)$ is continuous

R3. If $f = f * V_u(u) \in Y$ then $\pi(f)u \in S^*$

R4. $\pi(\overline{V_u(u)})u \in S$

and then define

$$\mathrm{Co}_S^u Y = \{v' \in S^* | V_u(v') \in Y\}$$

with norm $\|v'\| = \|V_u(v')\|_Y$

Remark

R1 and R4 hold automatically if we have a square integrable representation and $S = H$ or $S = H_\pi^\infty$.
R2 also holds for some non-square integrable representations.

A generalization (Coorbit spaces)

Theorem

- $\mathrm{Co}_S^u Y$ is a π-invariant Banach space
- $V_u : \mathrm{Co}_S^u Y \to Y * V_u(u) \subseteq Y$ is an isometric isomorphism

Theorem (Dependence on u and S)

- If (S, H, S^*) and (T, H, T^*) are Gelfand triples satisfying the assumptions then $\mathrm{Co}_S^u Y = \mathrm{Co}_T^u Y$ if $u \in S \cap T$.
- If u_1, u_2 satisfy assumptions and $V_{u_i}(v) * V_{u_j}(u_i) = C_{ij} V_{u_j}(v)$ and $Y \ni f \mapsto f * V_{u_i}(u_j) \in Y$ is continuous then $\mathrm{Co}_S^{u_1} Y = \mathrm{Co}_S^{u_2} Y$.

Remark The generalization for coorbit spaces also works if Y is a quasi-Banach space. It would be interesting to see if this can be used to describe the whole scale of Besov spaces.

Besov spaces revisited

It is possible to characterize the Besov spaces using the smooth vectors H_π^∞ in the following way:
Let G be the $ax+b$-group and π the quasi-regular representation on $L^2(\mathbb{R}^n)$. With ϕ as before

$$\dot{B}_s^{p,q} = \mathrm{Co}^{\phi}_{H_\pi^\infty} L^{p,q}_{s+n/2-n/q}$$

for $p,q \geq 1$.
By the previous theorems this follows from

$$S_0 \hookrightarrow H_\pi^\infty \hookrightarrow H_w^1$$

for $w(a,b) = a^{-s}$.
This has been verified for $n=1$.

Besov spaces for $1 \leq p, q \leq 2$

If $1 \leq p, q \leq 2$ and $s = n/q - n/2$ then $\pi(f)\phi \in H = L^2(\mathbb{R}^n)$. We therefore only need the Hilbert space H in the description, i.e.

$$\dot{B}^{p,q}_{n/q-n/2} = \mathrm{Co}^{\phi}_{H} L^{p,q}$$

This yields the (probably well known) fact that

$$\dot{B}^{p,q}_{n/q-n/2} \subseteq L^2(\mathbb{R}^n)$$

A note on Smooth vectors

Triebel has defined Sobolev spaces on Lie groups G.
If $X_1, \dots, X_n$ is a basis for $\mathfrak{g}$ then

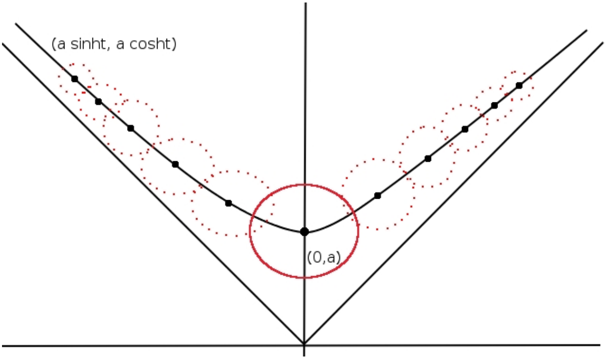
$$W_2^p = \left\{ f \;\middle|\; \sum_{i_1+\cdots+i_n \leq p} \|\pi(X_{j_1}^{i_1} \dots X_{j_k}^{i_n}) f\|_{L^2} < \infty \right\}$$

The smooth vectors can be characterized

$$H_\pi^\infty = \varinjlim \mathrm{Co}_H^\phi W_2^p$$

Besov Spaces on Cones

Let $\Lambda = \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 > 0\}$.



The group $G = \left\{ A(a,t) = \begin{pmatrix} a\cosh t & a\sinh t \\ a\sinh t & a\cosh t \end{pmatrix} \middle| a > 0, t \in \mathbb{R} \right\}$

acts transitively on Λ.

Besov Spaces on Cones

There is a Whitney decomposition of balls B_i of same size around points ξ_i and functions $\psi_i \in S_{\bar{\Lambda}}$ with

$$\mathrm{supp}(\widehat{\psi}_i) \subseteq B_i, \quad 0 \leq \widehat{\psi}_i, \quad \sum_i \widehat{\psi}_i = 1$$

The Besov spaces on the cone are defined by Békollé, Bonami, Garrigós, Ricci to be

$$B_s^{p,q} = \left\{ f \in \mathcal{S}'_{\bar{\Lambda}} \,\middle|\, \|f\|_{p,q,s} = \left(\sum_i \xi_i^{-s} \|f * \psi_i\|_p^q \right)^{1/q} < \infty \right\} / \mathcal{S}'_{\partial\Lambda}$$

The quasi-regular representation of $G \ltimes \mathbb{R}^2$ on $L^2_{\bar{\Lambda}}$

$$\pi(A,b)f(\xi) = \frac{1}{a} f(A^{-1}(\xi - b))$$

is square integrable (measure $\frac{da db dt}{a^{n+1}}$).

Non-unitary representations

Zimmermann has shown, that the non-unitary representation π of $SU(1,1)$ on $L^1(\mathbb{T})$ given by

$$\pi \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = |-\beta z + \bar{\alpha}|^{-2} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}} \right)$$

gives the orthogonality relation

$$\int \langle f_1, \pi^*(\alpha, \beta)\phi_1 \rangle \langle \pi(\alpha, \beta)\phi_2, f_2 \rangle dm(\alpha, \beta) = \langle f_1, f_2 \rangle \langle \phi_2, \phi_1 \rangle$$

for f_1, f_2, ϕ_1, ϕ_2 in appropriate spaces.
Perhaps this can replace the coorbit theory for homogeneous spaces [Dahlke, Steidl, Teschke]. .

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