Rado and Rado-type numbers

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Abstract

Ramsey theory is the study of the preservation of structure under finite partitions (called colorings). Ramsey theory on the integers studies particular structures (families of subsets of $\mathbb{Z}^+$) that are preserved under colorings of the integers (functions $\chi : \mathbb{Z}^+ \to \{1, 2, \ldots, r\}$ for some $r \in \mathbb{Z}^+$). A brief survey of Ramsey theory (focusing on its application to the integers) is presented, as well as several new results.

One such result is based on a theorem of R. Rado, which states that there exists a minimum integer N such that any 2-coloring of $\{1, 2, \ldots, N\}$ admits a monochromatic solution to any linear homogenous equation $\mathcal{E}$ with at least 3 variables and mixed-sign coefficients. Such an N is investigated for $x + y + kz = \ell w$ for $k, \ell \in \mathbb{Z}^+$, where N depends on k and ℓ. The exact such N is determined for $\ell - k \in \{0, 1, 2, 3, 4, 5\}$, for all k, ℓ for which $\frac{1}{2}((\ell - k)^2 - 2)(\ell - k + 1) \leq k \leq \ell - 4$, as well as for arbitrary k when $\ell = 2$.

Another result is that that, given two linear homogenous equations $\mathcal{E}_0, \mathcal{E}_1$, each with at least three variables and coefficients not all the same sign, any 2-coloring of $\mathbb{Z}^+$ admits monochromatic solutions of color 0 to $\mathcal{E}_0$ or monochromatic solutions of color 1 to $\mathcal{E}_1$. The 2-color off-diagonal Rado number $RR(\mathcal{E}_0, \mathcal{E}_1)$ is defined to be the smallest N such that $[1, N]$ must admit such solutions. A lower bound for $RR(\mathcal{E}_0, \mathcal{E}_1)$ is determined in certain cases when each $\mathcal{E}_i$ is of the form $a_1x_1 + \cdots + a_nx_n = z$, as is the exact value of $RR(\mathcal{E}_0, \mathcal{E}_1)$ when each is of the form $x_1 + a_2x_2 + \cdots + a_nx_n = z$.

0. Ramsey Theory

In Ramsey theory, combinatorics in general, and others branches of mathematics, there is a simple theorem that has proven to be an essential building block in several fundamental or important results: the pigeonhole principle. We first define an *r-coloring* of a set S as a function $\chi : S \to [r] = \{1, 2, \ldots, r\}$ for $r \in \mathbb{Z}^+$. Two elements s_1, s_2 of S are *monochromatic* when $\chi(s_1) = \chi(s_2)$ (this definition extends in the obvious way to more than two elements of S). We state the principle as follows:

Theorem 0.1 (The Pigeonhole Principle): For any set S, for any r-coloring of S where $r < |S|$, there are two monochromatic elements of S.

Proof: Consider, for contradiction, that $r < |S|$ but that some coloring χ does not give any color class with 2 elements. Then:

$$r < |S| = |\chi^{-1}(1)| + |\chi^{-1}(2)| + \cdots + |\chi^{-1}(r)| \leq 1 + 1 + \cdots + 1 = r$$

This contradiction immediately confirms the principle. Note that here $\chi^{-1}(i)$ denotes the i^{th} color class (elements of S assigned to the color i). □

While stated here in terms of a coloring (as suits a Ramsey theoretic context), it can be rephrased in other ways. For example, no function $f : [n] \to [m]$ can be injective if $n > m$. It is also immediately generalized so that $rn < |S|$ gives $n+1$ elements of the same color for any $n \in \mathbb{Z}^+$. The pigeonhole principle is attributed to Dirichlet [D]. The principle may also be used more generally. For example, if there are $a + b - 1$ elements of S to be 2-colored, either a are color 1 or b are color 2. Otherwise, there would be only $|\chi^{-1}(1)| + |\chi^{-1}(2)| \leq a - 1 + b - 1 = a + b - 2$ things in S.

0.1 Ramsey's Theorem

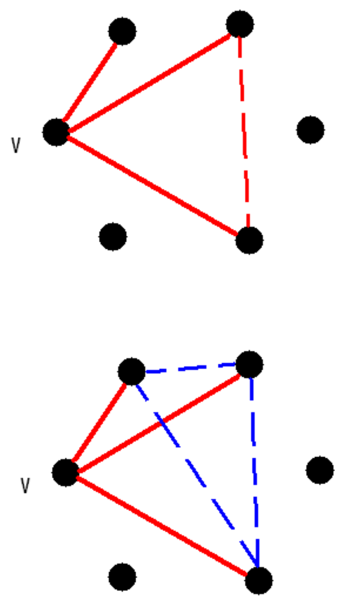
The pigeonhole principle is the key step in proving what is called Ramsey's theorem. We first introduce the concept of a graph. A graph is a set V of vertices and a set of pairs $E \subseteq V^2$ called edges. A graph is typically represented as points for V with segments between two points v_1, v_2 if $\{v_1, v_2\} \in V$. We say that a graph is complete on n vertices if $|V| = n$ and $E = V^2 - \{(i, i) : i \in V\}$ (it contains all possible edges between distinct vertices). Ramsey's theorem states:

Theorem 0.2 (Ramsey's theorem): For any $k, \ell \in \mathbb{Z}^+$ there exists an $n \in \mathbb{Z}^+$ such that for any 2-coloring of the edges of K_n, there is a monochromatic K_k of color 0 inside this K_n, or a monochromatic K_ℓ of color 1. Furthermore, the minimum such n, denoted $R(k, \ell)$.

This theorem was proven by Frank Ramsey in 1930 [Ram], and was the origin of this new branch of combinatorics. The canonical example of Ramsey's theorem is phrased as follows:

Theorem 0.2a (The party problem): If there are six guests at a party, and any pair of guests is either friends or strangers, then there is either a group of three who are mutual friends or a group of three who are mutual strangers.

Proof: We proceed by translating this into the language of graph theory. We define a graph on 6 vertices, each representing a person at the party. We have an edge between each pair, so the graph in question is K_6. Color an edge red between mutual friends and blue between mutual strangers (we often use red and blue instead of elements of [2]={1,2} for convenience). We seek a monochromatic K_3 (triangle).
Choose some vertex and apply the pigeonhole principle. Because there are 5 edges from this vertex, three must be the same color (without loss of generality, say red). Then consider the three vertices to which those edges connect. If any edge between the two is red, then we have a red K_3. Otherwise, they are all blue and we have a blue K_3. □



Two cases in this proof.

Proof of Theorem 0.2: The proof proceeds inductively on $k+\ell$ (with the case $k+\ell = 4$ trivial). Fix $k, \ell \in \mathbb{Z}^+$ and consider a complete graph on $R(k-1, \ell) + R(k, \ell-1)$ vertices with some coloring χ. Choose some vertex v and consider the subgraphs $B = \{x \in V : \chi(x, v) = 1\}$ (points with blue edges to v) and $R = V \backslash B$ (points with red edges to v). By the pigeonhole principle, either $|B| \geq R(k-1, \ell)$ or $|R| \geq R(k, \ell-1)$. If the former is true, then either B contains a red K_ℓ or a blue K_{k-1} (and by the definition of B, $B \cup \{v\}$ contains a blue K_k). The latter case is completely similar. □

Notice that our general proof involves finding a pre-existing K_{k-1}. This is hidden inside Theorem 0.2a; to construct our K_3 we find a pre-existing K_2 (note that K_2 is a single edge). Also be aware that Ramsey theory proofs almost universally make use of the well-ordering principle without stating it. Here, we've simply shown that *some* number exists such that K_n contains a blue K_k or a red K_ℓ. However, we did not mention that such a minimal n exists. It does, by the well-ordering principle (the set of all such n is a subset of $\mathbb{Z}^+$, which is well-ordered, and thus there is a minimal such n).

Ramsey's theorem is generalized naturally to allow for more than 2 colors, and to allow each color to have its own objective K_k. Precisely, we state:

Theorem 0.3 (Multicolored Ramsey's Theorem): For any r integers $k_1, k_2, \ldots, k_r \in \mathbb{Z}^+$, there is a number N such that any r-coloring of K_N admits a monochromatic K_{k_i} of the color k_i for some $i \in [r]$.

The implication is that structure is preserved. Under any coloring of the complete graph K_n we find smaller complete graphs with edges of the same color. We can find monochromatic K_k of any size k, provided we look in K_n for n sufficiently large.

This preservation of structure may run counter to intuition. Could there not be a way to achieve disorder on such a scale that monochromatic complete graphs are avoided? Indeed, this preservation of structure under colorings is prevalent in many mathematical contexts. There are a great many structures that simply cannot be avoided, or, in the words of T. Motzkin, "complete disorder is impossible" [JNR].

0.2 Ramsey Numbers

While Ramsey's theorem gives a constructive proof, it does so by using a graph that is taken to be purposely large. While we know that the Ramsey number $R(k, \ell)$ exists, it is not given by the theorem (the theorem gives an overestimate). So, although Ramsey's theorem tells us such a number exists, computing it precisely is a different matter, although it is a matter of particular interest.

0.3 Ramsey Theory

Ramsey theory is the study of the preservation of structure under colorings. This is a very general formulation, of course, so to give an idea of what Ramsey Theory encompasses, Graham, Rothschild, and Spencer (in their seminal work on Ramsey Theory [GRS]) gives the following list of 6 important Ramsey theoretic results:

Theorem 0.3 (Ramsey's theorem): As above.

Theorem 0.4 (van der Waerden's theorem): For any $\ell, r \in \mathbb{Z}^+$, there is an N such that for all $n \geq N$, any r-coloring of $[n]$ contains a monochromatic arithmetic progression of length ℓ.

Theorem 0.4 (Schur's theorem): For any $r \in \mathbb{Z}^+$, there is an N such that for all $n \geq N$, any r-coloring of $[n]$ contains a monochromatic solution (x, y, z) to $x + y = z$.

Theorem 0.6 (Rado's theorem for a single equation): For a homogeneous linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ and $r \in \mathbb{Z}^+$, there is an N such that for any $n \geq N$ any r-coloring of $[n]$ contains a monochromatic solution if and only if some (nonempty) subset of the coefficients c_i sums to zero.

Theorem 0.7 (Hales-Jewett theorem): For all $r, k \in \mathbb{Z}^+$, there is an N such that for $n \geq N$, and for any r-coloring of the elements of the n-dimensional cube of side-length k, written:

$$\{(x_1, x_2, \ldots, x_n) : x_i \in [k], i \in [n]\}$$

the coloring admits a monochromatic "line."

For $n = 2$, $r = 2$, and $k = 3$ we are looking at a filled tic-tac-toe board and hoping to find a winning "line" - although $n = 2$ is too small for such r, k, which is why tic-tac-toe ends in ties so frequently. In general, a "line" in this sense is a sequence of length k of the word $w(x)$ with letters from $[k]$, which contains some variable letter x taken to range over $[k]$. For example, if $w(x) = 1x$, then our line is 11,12,13. Note that this means the antidiagonal solution in tic-tac-toe (31,22,13) is not a "line" although the regular diagonal (11,22,33) is for $w(x) = xx$.

Theorem 0.8 (Graham-Leeb-Rothschild theorem): For a finite field F of q elements, and for any $k, \ell, r \in \mathbb{Z}^+$, there exists an N such that for all $n \geq N$, we have the following:

For any n-dimensional vector space over F, any r-coloring of the k-dimensional subspaces of V with r colors contains an ℓ dimensional subspace of V whose k-dimensional subspaces are monochromatic.

1. Ramsey theory meets the integers

Ramsey theory has applications to many mathematical structures: graphs, hypergraphs, groups, fields, vector spaces, and the integers. The focus of this thesis will be the integers.

1.1 Van der Waerden's Theorem

A seminal theorem in Ramsey theory on the integers is van der Waerden's theorem, conjectured by I. Schur and proven by B. L. van der Waerden [V]. It states that arithmetic progressions are one of the structures that are preserved under colorings.

Theorem 1.1 (van der Waerden's theorem): For any $k, r \in \mathbb{Z}^+$, there is an N such that for all $n \geq N$, any r-coloring of $[n]$ contains a monochromatic arithmetic progression of length k. The minimal such N is denoted $w(k, r)$.

The proof of this theorem ([GR]), like that of Ramsey's theorem, provides an upper bound on $w(k, r)$. This bound is exceedingly large - well beyond known values and open to substantial improvement. Gowers, for example, has improved these bounds [Go]. It is reasonable to say that while an arithmetic progression must exist within the interval $[w(k, r)]$, it is easier to prove that an arithmetic progression exists in an interval $[N]$ much longer than $[w(k, r)]$, hence the abundance of large upper bounds in such problems.

1.2 Equation Regularity and Schur's Theorem

Besides geometrically defined structures (graphs of particular shapes, arithmetic progressions, etc.), the family of solutions to a particular equation may (or may not) be preserved under colorings. For $r \in \mathbb{Z}^+$, an equation $\mathcal{E}$ is said to be r-*regular* (resp. regular) if it contains monochromatic solutions under any r-coloring (resp. any finite coloring) of $\mathbb{Z}^+$.

An obvious observation tells us that an r-regular equation $\mathcal{E}$ must have infinitely many solutions. However, this is not sufficient. For example $x = 2y$ is not regular or even 2-regular. To see this, let $d_2(n)$ be the number of times 2 evenly divides n. Coloring each integer n based on the parity of $d_2(n)$ provides a coloring admitting no monochromatic solutions to $x = 2y$. Note that regularity implies r regularity for any r, as does $(r + 1)$-regularity

The question becomes "which equations are regular or r-regular?" This question is, for the most part, completely unknown. Indeed, many diophantine equations are themselves not well understood. We will discover that the regularity of simple diophantine equations (particularly those that are linear) is open to fruitful investigation. This question has been considered for some time, since the 1940s-1950s, however there are many results predating Ramsey theory that are relevant.

Schur's theorem is simply the statement that the equation $x+y=z$ is regular. It is arguably the first Ramsey theoretic result, having been proven in 1916 (predating Ramsey's theorem). However, the simplest proof of this theorem makes use of Ramsey's theorem as follows.

Theorem 1.2 (Schur's theorem): The equation $x+y=z$ is regular.

Proof [GRS]: Fix $r \in \mathbb{Z}^+$ and take $n+1 \geq R(\overbrace{3,3,\ldots,3}^{r})$ (recall that this is a Ramsey number for r colors). Then an r-coloring χ of $[n]$ induces an r-coloring χ^* of the graph K_{n+1} on $\{0,1,\ldots,n\}$ where the color of an edge is $\chi^*(i,j) = \chi(|i-j|)$. Then there must be a monochromatic triangle $(i,j),(j,k),(k,i)$. This corresponds to the monochromatic solution $(x,y,z) = (i-j, j-k, i-k)$ $(x,y,z>0$ without loss of generality). $\square$

Schur proved his theorem in a paper on solutions to the equation $x^m + y^m \equiv z^m \pmod{p}$, and he did not explore the possible generalizations of this theorem. His student, Richard Rado, would do so, but neither Schur nor Rado would phrase their results in terms of Ramsey theory. As a field of study, Ramsey theory came about later, years after Ramsey originally proved his theorem. Schur's theorem was motivated by an investigation into Fermat's conjecture (as one might guess), and the Ramsey theoretic implications were not obvious and only more recently received thorough investigation.

Rado, in continuing Schur's work, would characterize the regularity of any homogeneous linear equation (indeed, any system of such equations) as follows.

Theorem 1.3a (Rado's theorem, single equation) [Rad]: A homogeneous linear equation $c_1x_1 + \cdots + c_nx_n = 0$ is regular if and only if some subset of its coefficients sum to 0.

Theorem 1.3b (Rado's theorem) [Rad]: A system $\mathcal{L}$ of linear equations written $\mathbf{A}x = \mathbf{0}$ is regular if and only if $\mathbf{A}$ meets the columns condition. The columns condition requires that the columns of A can be partitioned into sets of columns $C_1, \ldots, C_n$ such that the sum over C_1 is $\mathbf{0}$, and the sum over each subsequent C_i is in the span of $C_1, \ldots, C_{i-1}$.

For example, consider the system:

$$x - y + 2w + 3v = 0$$

$$-2x + y + z - 3w - 3v = 0$$

$$x + y - 2z - 2w - v = 0$$

This system meets the column's condition. The first three columns sum to zero:

$$[1,-2,1] + [-1,1,1] + [0,1,-2] = [0,0,0]$$

The sum of the other two columns is in the span of the first three:

$$[2,-3,-2]+[3,-3,-1]=[5,-6,-3]$$

$$2[1,-2,1]-3[-1,1,1]+[0,1,-2]=[5,-6,-3]$$

It is clear that Theorem 1.3b is a generalization of 1.3a, since the subset C_1 that sums to zero corresponds to the subset mentioned in 1.3a, and for a single equation, columns are simply integers, any of which are in the span of C_1.

Rado proved the general case first; the single-equation version is just a simpler special case. The reader is directed to [GRS] for proof of either version. The ambitious reader is directed to [R] for the original proof of theorem 1.3b (in German).

Rado's theorem leads us to new questions. What other kinds of equations are regular? What kinds of equations are 2-regular (or r regular for some $r \in \mathbb{Z}^+$) but not regular? What are the Ramsey-type numbers associated with particular equations (called, unsurprisingly, Rado numbers). This paper will now focus on determining the Rado numbers for particular sorts of linear equations, and will also prove that *off-diagonal Rado numbers* exist for those same equations (and subsequently, investigate these off-diagonal Rado numbers). For a broad survey of Ramsey theoretic results on the integers, the reader is directed to [LR]. Other results are documented in [Gr2] and [KL].

1.3 The Compactness Principle

Within this paper (as elsewhere), we will often transition between two types of Ramsey theoretic statements: the finitary and infinitary versions of theorems. For example, we may state van der Waerden's theorem as above (referring to some N such that we find an arithmetic progression of the required length), but we may also simply refer to a coloring of $\mathbb{Z}^+$ in its entirety, which admits arbitrarily long (but not necessarily infinite) arithmetic progressions. Certainly, making $n \geq N$ larger is required for larger k (i.e., $w(k;r)$ is increasing in k, clearly), but it is not so trivial to leap to the infinitary version (the converse, however, is trivial).

Theorem 1.4 (The Compactness Principle): Let $H=(V,E)$ be a hypergraph where members of E are finite. Suppose for all finite $W \subseteq V$ that $\chi(H_W) \leq r$ for some $r \in \mathbb{Z}^+$ (H_W is the restriction of the graph H to the vertices in W). Then $\chi(H) \leq r$.

Here χ represents the chromatic number of the graph. It is (in essence) the Ramsey-type number from the hypergraph perspective. Often, the principle is phrased in terms of the contrapositive: if $\chi(H) > r$ then there exists a finite W with $\chi(H_W) > r$. In any event, we may consider vertices of the hypergraph to be $\mathbb{Z}^+$, and edges to be the Ramsey-type structure in question. Proof is omitted, but can be found in [GRS].

2. 2-color Rado Numbers

In addition to his eponymous theorem, Rado also proved the following, lesser known, result.

Theorem 2.1 [Rad]: Let $\mathcal{E} = 0$ be a linear homogeneous equation with integer coefficients. Assume that $\mathcal{E}$ has at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits a monochromatic solution to $\mathcal{E} = 0$.

Theorem 2.1 cannot be extended to more than 2 colors, without further restriction on the equation. For example, Fox and Radoičić [FR] have shown, in particular, that there exists a 3-coloring of $\mathbb{Z}^+$ that admits no monochromatic solution to $x + 2y = 4z$. For more information about equations that have finite colorings of $\mathbb{Z}^+$ with no monochromatic solution see [AFG] and [FR]. This means that there is a fundamental difference between 2-regularity and regularity or even between 2- and 3-regularity.

In essence, any (sensible) homogenous linear equation meets the conditions of this theorem. Here, sensible essentially means that it has solution in $\mathbb{Z}^+$ (e.g., not $x + y = 0$) and that its solutions are at least slightly interesting. The solutions to $5x - 7y = 0$, for example, can almost immediately be 2-colored to avoid admitting monochromatic solutions. Color an integer red if 5 has even multiplicity in its prime factorization, and blue otherwise (this is very similar to the example in Section 1.2). There are equations that are 2-regular that don't meet the conditions of the theorem (e.g., $x - y = 0$).

Proof [Rad]: Let $\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{\ell} \beta_i y_i$ be our equation, where $k \geq 2$, $\ell \geq 1$, $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$, and $\beta_i \in \mathbb{Z}^+$ for $1 \leq i \leq \ell$. By setting $x = x_1 = x_2 = \cdots = x_{k-1}$, $y = x_k$, and $z = y_1 = y_2 = \cdots = y_\ell$, we may consider solutions to

$$ax + by = cz,$$

where $a = \sum_{i=1}^{k-1} \alpha_i$, $b = c_k$, and $c = \sum_{i=1}^{\ell} \beta_i$. We will denote $ax + by = cz$ by $\mathcal{E}$.

Let $m = \operatorname{lcm}\left(\frac{a}{\gcd(a,b)}, \frac{c}{\gcd(b,c)}\right)$. Let (x_0, y_0, z_0) be the solution to $\mathcal{E}$ with $\max(x, y, z)$ a minimum, where the maximum is taken over all solutions of positive integers to $\mathcal{E}$. Let $A = \max(x_0, y_0, z_0)$.

Assume, for a contradiction, that there exists a 2-coloring of $\mathbb{Z}^+$ with no monochromatic solution to $\mathcal{E}$. First, note that for any $n \in \mathbb{Z}^+$, the set $\{in : i = 1, 2, \ldots, A\}$ cannot be monochromatic, for otherwise $x = x_0 n$, $y = y_0 n$, and $z = z_0 n$ is a monochromatic solution, a contradiction.

Let $x = m$ so that $\frac{bx}{a}, \frac{bx}{c} \in \mathbb{Z}^+$. Letting red and blue be our two colors, we may assume, without loss of generality, that x is red. Let y be the smallest number in $\{im : i = 1, 2, \ldots, A\}$ that is blue. Say $y = \ell m$ so that $2 \leq \ell \leq A$.

For some $n \in \mathbb{Z}^+$, we have that $z = \frac{b}{a}(y-x)n$ is blue, otherwise $\{i\frac{b}{a}(y-x) : i = 1, 2, \ldots\}$ would be red, admitting a monochromatic solution to $\mathcal{E}$. Then $w = \frac{a}{c}z + \frac{b}{c}y$ must be red, for otherwise $az + by = cw$ and z, y, and w are all blue, a contradiction. Since x and w are both red, we have that $q = \frac{c}{a}w - \frac{b}{a}x = \frac{b}{a}(y-x)(n+1)$ must be blue, for otherwise x, w, and q give a red solution to $\mathcal{E}$. As a consequence, we see that $\left\{i\frac{b}{a}(y-x) : i = n, n+1, \ldots\right\}$ is monochromatic. This gives us that $\left\{i\frac{b}{a}(y-x)n : i = 1, 2, \ldots, A\right\}$ is monochromatic, a contradiction. $\square$

In this thesis we study the equation $x + y + kz = \ell w$ for positive integers k and ℓ. As such, we make the following notation.

Notation: For k a positive integer and $j > -k$ an integer, let $\mathcal{E}(k, j)$ represent the equation

$$x + y + kz = (k + j)w.$$

2.1 A General Upper Bound

Definition: Let $\mathcal{E}$ be any equation that satisfies the conditions in Theorem 1. Denote by $RR(\mathcal{E})$ the minimum integer N such that any 2-coloring of $[N]$ admits a monochromatic solution to $\mathcal{E}$.

Our goal is to investigate these Rado numbers for the equations in question. Theorems to this end most frequently come in two varieties: upper and lower bounds. To show that some N is a lower bound, we find some particular coloring of $[N-1]$ that does not admit monochromatic solutions. Conversely, to show that N' is an upper bound, we show that any coloring of $[N']$ must admit a monochromatic solution. If $N = N'$, then the $RR(\mathcal{E}) = N$.

Part of the following result is essentially a result due to Burr and Loo [BL] who show that, for $j \geq 4$, we have $RR(x + y = jw) = \binom{j+1}{2}$. This result was never published. Below is an (independently derived) proof.

Theorem 2.2: Let $k, j \in \mathbb{Z}^+$ with $j \geq 4$. Then $RR(\mathcal{E}(k, j)) \leq \binom{j+1}{2}$. Furthermore, for all $k \geq \frac{(j^2-2)(j+1)}{2}$, we have $RR(\mathcal{E}(k, j)) = \binom{j+1}{2}$.

So to prove this theorem, we will establish $\binom{j+1}{2}$ as an upper bound for any k, and then show that it is also a lower bound for k sufficiently large.

Proof: Let $\mathcal{F}$ denote the equation $x + y = jw$. We will show that $RR(\mathcal{F}) \leq \binom{j+1}{2}$. Since any solution to $\mathcal{F}$ is also a solution to $\mathcal{E}(k, j)$ for any $k \in \mathbb{Z}^+$, the first statement will follow.

Assume, for a contradiction, that there exists a 2-coloring of $\left[1, \binom{j+1}{2}\right]$ with no monochromatic solution to $\mathcal{F}$. Using the colors red and blue, we let R be the set of red integers and

B be the set of blue integers. We denote solutions of $\mathcal{F}$ by (x, y, w) where $x, y, w \in \mathbb{Z}^+$.

Since $(x, y, w) = (j-1, 1, 1)$ solves $\mathcal{F}$, we may assume that $1 \in R$ and $j-1 \in B$. We separate the proof into two cases.

`Case 1.` $j+1 \in B$. Assume $i \geq 1$ is red. Considering $(1, ij-1, i)$ gives $ij-1 \in B$. If $i \leq \left\lfloor \frac{j+1}{2} \right\rfloor$, this, in turn, gives us $i+1 \in R$ by considering $(ij-1, j+1, i+1)$. Hence $1, 2, \ldots, \left\lfloor \frac{j+1}{2} \right\rfloor + 1$ are all red. But then $\left(\left\lfloor \frac{j}{2} \right\rfloor, \left\lfloor \frac{j+1}{2} \right\rfloor, 1 \right)$ is a red solution, a contradiction.

`Case 2.` $j+1 \in R$. This implies that $j\left(\frac{j+1}{2}\right) \in B$. Note also that the solutions $(1, j-1, 1)$ and $\left(j\left(\frac{j-1}{2}\right), j\left(\frac{j-1}{2}\right), j-1\right)$ give us $j-1 \in B$ and $j\left(\frac{j-1}{2}\right) \in R$.

First consider the case when j is even. By considering $\left(\frac{j}{2}, \frac{j}{2}, 1\right)$ we see that $\frac{j}{2} \in B$. Assume, for $i \geq 1$, that $\frac{(2i-1)j}{2} \in B$. Considering $\left(j\left(\frac{j+1}{2}\right), \frac{(2i-1)j}{2}, \frac{j}{2}+i\right)$ we have $\frac{j}{2}+i \in R$. This, in turn, implies that $\frac{(2i+1)j}{2} \in B$ by considering $\left(j\left(\frac{j-1}{2}\right), \frac{(2i+1)j}{2}, \frac{j}{2}+i\right)$. Hence we have $\frac{j}{2}+i$ is red for $1 \leq i \leq \frac{j}{2}$. This gives us that $j-1 \in R$ (when $i = \frac{j}{2}-1$), a contradiction.

Now consider the case when j is odd. We consider two subcases.

`Subcase i.` $j \in B$. For $i \geq 1$, assume that $ij \in B$. We obtain $\frac{j+1}{2}+i \in R$ by considering the solution $\left(j\left(\frac{j+1}{2}\right), ij, \frac{j+1}{2}+i\right)$. This gives us $(i+1)j \in B$ by considering $\left(j\left(\frac{j-1}{2}\right), (i+1)j, \frac{j+1}{2}+i\right)$ Hence, we have that $j, 2j, \ldots, \left(\frac{j+1}{2}\right)j$ are all blue, contradicting the deduction that $j\left(\frac{j-1}{2}\right) \in R$.

`Subcase ii.` $j \in R$. We easily have $2 \in B$. Next, we conclude that $j\left(\frac{j-3}{2}\right) \in R$ by considering $\left(j\left(\frac{j+1}{2}\right), j\left(\frac{j-3}{2}\right), j-1\right)$. Then, the solution $\left(j\left(\frac{j-3}{2}\right), j\left(\frac{j-1}{2}\right), j-2\right)$ gives us $j-2 \in B$. We use $\left(j\left(\frac{j-1}{2}\right), j, \frac{j+1}{2}\right)$ to see that $\frac{j+1}{2} \in B$. From $\left(j\left(\frac{j+1}{2}\right)-2, 2, \frac{j+1}{2}\right)$ we have $j\left(\frac{j+1}{2}\right)-2 \in R$. To avoid $\left(j\left(\frac{j-1}{2}\right)+2, j\left(\frac{j+1}{2}\right)-2, j\right)$ being a red solution, we have $j\left(\frac{j-1}{2}\right)+2 \in B$. This gives us a contradiction; the solution $\left(j\left(\frac{j-1}{2}\right)+2, j-2, \frac{j+1}{2}\right)$ is blue.

This completes the proof of the first statement of the theorem.

For the proof of the second statement of the theorem, we need only provide a lower bound of $\binom{j+1}{2}-1$. We first show that any solution to $x+y+kz = (k+j)w$ with $x, y, z, w < \binom{j+1}{2}$ must have $z = w$ when $k \geq \frac{(j^2-2)(j+1)}{2}$. Assume, for a contradiction, that $z \neq w$. If $z < w$, then $(k+j)w \geq (k+j)(z+1) > kz+k$. However, $x+y < j(j+1)$ while $k > j(j+1)$ for $j \geq 3$. Hence, $z \not< w$. If $z > w$, then $(k+j)w \leq k(z-1) + j\left(\binom{j+1}{2}-1\right)$. Since we have $x+y+kz = (k+j)w$ we now have $2+kz \leq x+y+kz \leq k(z-1)+j\left(\binom{j+1}{2}-1\right)$. Hence, $k \leq j\left(\binom{j+1}{2}-1\right)-2$, contradicting the given bound on k. Thus, $z = w$.

Now, any solution to $x+y+kz = (k+j)w$ with $z = w$ is a solution to $x+y = jw$. From

Burr and Loo's result, there exists a 2-coloring of $\left[1, \binom{j+1}{2} - 1\right]$ with no monochromatic solution to $x + y = jw$. This provides us with a 2-coloring with no monochromatic solution to $x + y + kz = (k + j)w$, thereby finishing the proof of the second statement. $\square$

This tells us that for a fixed $j = \ell - k \geq 4$, the Rado number for $x+y+kz = \ell w$ is eventually constant, which would follow along the diagonals of some table of k vs. ℓ (such as Table 1, presented below). We turn to the exact Rado numbers for particular small values of j.

2.2 Some Specific Numbers

In this section we determine the exact values for $RR(\mathcal{E}(k, j))$ for $j \in \{0, 1, 2, 3, 4, 5\}$, most of which are cases not covered by Theorem 2.

Theorem 2.3: For $k \geq 2$,

$$RR(\mathcal{E}(2k, 0)) = 2k \text{ and } RR(\mathcal{E}(2k - 1, 0)) = 3k - 1.$$

Furthermore $RR(\mathcal{E}(2, 0)) = 5$ and $RR(\mathcal{E}(1, 0)) = 11$.

Proof: The cases $RR(\mathcal{E}(2, 0)) = 5$, $RR(\mathcal{E}(4, 0)) = 4$, and $RR(\mathcal{E}(3, 0) = 5$ are easy calculations, as is $RR(\mathcal{E}(1, 0)) = 11$, which first appeared in [BB]. Hence, we may assume $k \geq 3$ in the following arguments.

We start with $RR(\mathcal{E}(2k, 0)) = 2k$. To show that $RR(\mathcal{E}(2k, 0)) \geq 2k$ consider the 2-coloring of $[1, 2k - 1]$ defined by coloring the odd integers red and the even integers blue. To see that there is no monochromatic solution to $x + y + 2kz = 2kw$, note that we must have $2k \mid (x + y)$. This implies that $x + y = 2k$ since $x, y \leq 2k - 1$. Thus, $w = z + 1$. However, no 2 consecutive integers have the same color. Hence, any solution to $\mathcal{E}(2k, 0)$ is necessarily bichromatic.

Next, we show that $RR(\mathcal{E}(2k, 0)) \leq 2k$. Assume, for a contradiction, that there exists a 2-coloring of $[1, 2k]$ with no monochromatic solution to our equation. Using the colors red and blue, we may assume, without loss of generality, that k is red. This gives us that $k - 1$ and $k+1$ are blue, by considering $(x, y, z, w) = (k, k, k-1, k)$ and $(k, k, k, k+1)$. Using these in the solution $(2k, 2k, k - 1, k + 1)$ we see that $2k$ must be red, which implies that $k - 2$ is blue (using $(2k, 2k, k-2, k)$). However, this gives the blue solution $(k-1, k+1, k-2, k-1)$, a contradiction.

We move on to $RR(\mathcal{E}(2k - 1, 0))$. To show that $RR(\mathcal{E}(2k - 1, 0)) \leq 3k - 1$ consider the following 2-colorings of $[1, 3k - 2]$, dependent on k (we use r/b for red/blue, respectively):

$$\begin{array}{ll} brrbbrrbb\ldots bbr & \text{if } k\equiv 0 \pmod 4\\ rrbbrrbb\ldots bbr & \text{if } k\equiv 1 \pmod 4\\ brrbbrrbb\ldots rrb & \text{if } k\equiv 2 \pmod 4\\ rrbbrrbb\ldots rrb & \text{if } k\equiv 3 \pmod 4. \end{array}$$

Since we need $(2k-1) \mid (x+y)$ and $x,y \leq 3k-2$, we have $x+y \in \{2k-1, 4k-2\}$. By construction, if $x+y=2k-1$, then x and y have different colors. Hence, the only possibility is $x+y=4k-2$. But then $w=z+2$ and we see that w and z must have different colors.

Next, we show that $RR(\mathcal{E}(2k-1,0)) \leq 3k-1$. Assume, for a contradiction, that there exists a 2-coloring of $[1,3k-1]$ with no monochromatic solution to our equation. Using the colors red and blue, we may assume, without loss of generality, that $2k-1$ is red. To avoid $(2k-1,2k-1,z,z+2)$ being a red solution, we see that $2k+1$ and $2k-3$ are blue (using $z=2k-1$ and $2k-3$, respectively).

If $2k$ is red, then $2k-2$ is blue (using $(2k-2,2k,2k-2,2k)$). From $(3k-1,3k-1,2k-2,2k+1)$ we see that $3k-1$ is red. This, in turn, implies that $2k-4$ is blue (using $(3k-1,3k-1,2k-4,2k-1)$). So that $(2k-4,2k+2,2k-4,2k-2)$ is not a blue solution, we require $2k+2$ to be red. But then $(2k-1,2k-1,2k,2k+2)$ is a red solution, a contradiction.

If $2k$ is blue, then $2k-2$ must be red. So that $(3k-1,3k-1,2k-3,2k)$ is not a blue solution, we have that $3k-1$ is red. Also, $2k+2$ must be red by considering $(2k-3,2k+1,2k,2k+2)$. But this implies that $(3k-1,3k-1,2k-1,2k+2)$ is a red solution, a contradiction. □

We proceed with a series of results for the cases $j=1,3,4,5$. When $j=2$, the corresponding number is trivially 1 for all $k\in\mathbb{Z}^+$.

Below, we will call a coloring of $[1,n]$ *valid* if it does not contain a monochromatic solution to $\mathcal{E}(k,j)$.

Theorem 2.4: For $k\in\mathbb{Z}^+$,

$$RR(\mathcal{E}(k,1)) = \begin{cases} 4 & \text{for } k \leq 3\\ 5 & \text{for } k \geq 4. \end{cases}$$

Proof: Assume, for a contradiction, that there exists a 2-coloring of $[1,5]$ with no monochromatic solution to $x+y+kz=(k+1)w$. We may assume that 1 is red. Considering the solutions $(1,1,2,2),(2,2,4,4),(1,3,4,4)$, and $(2,3,5,5)$, in order, we find that 2 is blue, 4 is red, 3 is blue, and 5 is red. But then $(1,4,5,5)$ is a red solution, a contradiction. Hence, $RR(\mathcal{E}(k,1)) \leq 5$ for all $k\in\mathbb{Z}^+$.

We see from the above argument that the only valid colorings of $[1, 3]$ (assuming, without loss of generality, that 1 is red) are rbr and rbb (where we use r for red and b for blue). Furthermore, the only valid coloring of $[1, 4]$ is $rbbr$. We use these colorings to finish the proof.

First consider the valid coloring rbr. The possible values of $x+y+kz$ when x, y, z are all red form the set $\{k+2, k+4, k+6, 3k+2, 3k+4, 3k+6\}$. The possible values when x, y, z are all blue is $2k+4$. The possible values of $(k+1)w$ when w is red form the set $\{k+1, 3k+3\}$; when w is blue, $2k+2$ is the only possible value. We denote these results by:

$$\begin{aligned} R_{x,y,z} &= \{k+2, k+4, k+6, 3k+2, 3k+4, 3k+6\} \\ B_{x,y,z} &= \{2k+4\} \\ R_w &= \{k+1, 3k+3\} \\ B_w &= \{2k+2\}. \end{aligned}$$

Next, we determine those values of k, if any, for which $R_{x,y,z} \cap R_w \neq \emptyset$ or $B_{x,y,z} \cap B_w \neq \emptyset$. Clearly, there is no such k for these sets. Hence, we conclude that $RR(\mathcal{E}(k, 1)) \geq 4$ for all k. (We need not consider the valid coloring rbb since we now know that $RR(\mathcal{E}(k, 1)) \geq 4$ for all k.)

We move on to the valid coloring of $[1, 4]$, which is $rbbr$. We find that

$$\begin{aligned} R_{x,y,z} &= \{k+2, k+5, k+8, 4k+2, 4k+5, 4k+8\} \\ B_{x,y,z} &= \{2k+4, 2k+5, 2k+6, 3k+4, 3k+5, 3k+6\} \\ R_w &= \{k+1, 4k+4\} \\ B_w &= \{2k+2, 3k+3\}. \end{aligned}$$

We see that $B_{x,y,z} \cap B_w \neq \emptyset$ when $k = 1$ $(2k+4 = 3k+3)$, $k = 2$ $(2k+5 = 3k+3)$, and $k = 3$ $(2k+6 = 3k+3)$. For all other values of k, $B_{x,y,z} \cap B_w = \emptyset$ and $R_{x,y,z} \cap R_w = \emptyset$. Hence, we conclude that $RR(\mathcal{E}(k, 1)) \geq 5$ for $k \geq 3$, while, since $rbbr$ is the only valid coloring of $[1, 4]$, $RR(\mathcal{E}(k, 1)) \leq 4$ for $k = 1, 2, 3$. This completes the proof of the theorem. □

2.2a About FVR

The proofs below refer to the small Maple package `FVR`. We find our lower bounds by considering all valid colorings of $[1, n]$ for some $n \in \mathbb{Z}^+$ and deducing the possible elements that $x + y + kz$ can be when x, y, and z are monochromatic and the possible elements that $(k+j)w$ can be, i.e., determining $R_{x,y,z}$, $B_{x,y,z}$, R_w, and B_w. We then looked for intersections that would make (x, y, z, w) a monochromatic solution. The intersections are specific values of k which show that the given coloring has monochromatic solutions for these values of k.

The Maple package `FVR` automates this process in, and is available from Aaron Robertson's webpage[1]. The input is a list of all valid colorings of $[1, n]$. The output is a list of values of k for which we have monochromatic solutions. By increasing n we are able to determine the exact Rado numbers for all $k \in \mathbb{Z}^+$. An example of this is explained in detail in the proof of the next theorem.

Theorem 2.5: For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,3)) = \begin{cases} 4 & \text{for } k \leq 5 \text{ and } k = 7 \\ 6 & \text{for } k = 8, 11 \\ 9 & \text{for } k = 6, 9, 10 \text{ and } k \geq 12 \end{cases}$$

Proof: The method of proof is the same as that for Theorem 4, but we will work it out in some detail commenting on the use of the Maple package `FVR`.

It is easy to check that the only valid 2-colorings (using r for red, b for blue, and assuming that 1 is red) of $[1, n]$ for $n = 4, 5, \ldots, 8$ are as in the following table. The determinations of $R_{x,y,z}, R_w, B_{x,y,z}$, and B_w are equally easy.

n	coloring	sets
4	$rbrr$	$R_{x,y,z} = \{ik + j : i = 1, 3, 4; j = 2, 4, 5, 6, 7, 8\}; R_w = \{i(k + 3) : i = 1, 3, 4\}$
		$B_{x,y,z} = \{2k + 2\}; B_w = \{2k + 6\}$
5	$rbrrb$	$R_{x,y,z} = \{ik + j : i = 1, 3, 4; j = 2, 4, 5, 6, 7, 8\}; R_w = \{i(k + 3) : i = 1, 3, 4\}$
		$B_{x,y,z} = \{ik + j : i = 2, 5; j = 2, 4, 7, 10\}; B_w = \{2k + 6, 5k + 15\}$
6	$rbrrbb$	$R_{x,y,z} = \{ik + j : i = 1, 3, 4; j = 4, 5, 6, 7, 8\}; R_w = \{i(k + 3) : i = 1, 3, 4\}$
		$B_{x,y,z} = \{ik + j : i = 2, 5, 6; j = 4, 7, 8, 10, 11, 12\}; B_w = \{2k + 6, 5k + 15, 6k + 18\}$
7	$rbrrbbr$	$R_{x,y,z} = \{ik + j : i = 1, 3, 4, 7; j = 4, \ldots, 8, 10, 11, 14\}; R_w = \{i(k + 3) : i = 1, 3, 4, 7\}$
		$B_{x,y,z} = \{ik + j : i = 2, 5, 6; j = 4, 7, 8, 10, 11, 12\}; B_w = \{2k + 6, 5k + 15, 6k + 18\}$
8	$rbrrbbrb$	$R_{x,y,z} = \{ik + j : i = 1, 3, 4, 7; j = 4, \ldots, 8, 10, 11, 14\}; R_w = \{i(k + 3) : i = 1, 3, 4, 7\}$
		$B_{x,y,z} = \{ik + j : i = 2, 5, 6; j = 4, 7, 8, 10, \ldots, 14, 16\}; B_w = \{i(k + 3) : i = 2, 5, 6, 8\}$

The sets $R_{x,y,z}, R_w, B_{x,y,z}$, and B_w are automatically found by `FVR`, which then gives us the values of k that induce a nonempty intersection of either $R_{x,y,z} \cap R_w$ or $B_{x,y,z} \cap B_w$. For completeness, we give the details.

For the coloring $rbrr$, we have $R_{x,y,z} \cap R_w \neq \emptyset$ when $k = 1$ $(k + 3 = 3k + 5)$, $k = 2$ $(3k + 9 = 4k + 7)$, $k = 3$ $(3k + 9 = 4k + 6)$, $k = 4$ $(3k + 9 = 4k + 5)$, $k = 5$ $(3k + 9 = 4k + 4)$,

[1]`http://math.colgate.edu/∼aaron/programs.html`

and $k = 7$ $(3k + 9 = 4k + 2)$. Since *rbrr* is the only valid coloring of $[1, 4]$, we have $RR(\mathcal{E}(k, 3)) = 4$ for $k = 1, 2, 3, 4, 5, 7$.

For the coloring *rbrrb*, we have no new additional elements in $R_{x,y,z} \cap R_w$. Hence, any possible additional intersection point comes from $B_{x,y,z} \cap B_w$. However, $B_{x,y,z} \cap B_w = \emptyset$ for all $k \in \mathbb{Z}^+$. Hence, $RR(\mathcal{E}(k, 3)) \geq 6$ for $k \in \mathbb{Z}^+ \setminus \{1, 2, 3, 4, 5, 7\}$.

For *rbrrbb*, we again have no new additional elements in the red intersection. We do, however, have additional elements in $B_{x,y,z} \cap B_w$. When $k = 8$ $(5k + 15 = 6k + 7)$ and $k = 11$ $(5k + 15 = 6k + 4)$ we have a blue intersection. We conclude that $RR(\mathcal{E}(k, 3)) = 6$ for $k = 8, 11$ and $RR(\mathcal{E}(k, 3)) \geq 7$ for $k \in \mathbb{Z}^+ \setminus \{1, 2, 3, 4, 5, 7, 8, 11\}$.

Considering *rbrrbbr*, we have no new additional elements in $B_{x,y,z} \cap B_w$. Furthermore, we have no new additional intersection points in $R_{x,y,z} \cap R_w$. Hence, $RR(\mathcal{E}(k, 3)) \geq 8$ for $k \in \mathbb{Z}^+ \setminus \{1, 2, 3, 4, 5, 7, 8, 11\}$.

Lastly, we consider *rbrrbbrb*, which gives no new additional elements in $R_{x,y,z} \cap R_w$. Furthermore, we have no new additional intersection points in $B_{x,y,z} \cap B_w$. Thus, $RR(\mathcal{E}(k, 3)) \geq 9$ for $k \in \mathbb{Z}^+ \setminus \{1, 2, 3, 4, 5, 7, 8, 11\}$.

Analyzing the valid coloring of $[1, 8]$ we see that we cannot extend it to a valid coloring of $[1, 9]$. Hence, $RR(\mathcal{E}(k, 3)) \leq 9$ for all k so that $RR(\mathcal{E}(k, 3)) = 9$ for $k \in \mathbb{Z}^+ \setminus \{1, 2, 3, 4, 5, 7, 8, 11\}$. $\square$

Theorem 2.6: For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k, 4)) = \begin{cases} 3 & \text{for } k = 2, 3, 4 \\ 5 & \text{for } k = 6, 7, 8, 10, 11, 14 \\ 6 & \text{for } k = 5, 9, 12, 13, 15, 18 \\ 8 & \text{for } k = 17, 19, 22 \\ 9 & \text{for } k = 1, 23, 24 \\ 10 & \text{for } k = 16, 20, 21 \text{ and } k \geq 25. \end{cases}$$

Proof: Use the Maple package `FVR` with the following valid colorings (which are easily obtained):

n	valid colorings
3	rbb
4	$rbbr$
5	$rbbrr$
6	$rbbrrr$
7	$rbbrrrr, rbbrrrb$
8	$rbbrrrrb, rbbrrrbb$
9	$rbbrrrrbr, rbbrrrbbr$
10	none

Note that if $[1, n]$ has more than one valid coloring, we can conclude that $RR(\mathcal{E}(k, 4)) \leq n$ for $k = \hat{k}$ only if $\hat{k}$ is an intersection point for *all* valid colorings. Otherwise, there exists a coloring of $[1, n]$ that avoids monochromatic solutions to $\mathcal{E}(k, 4)$ when $k = \hat{k}$. $\square$

Theorem 2.7: For $k \in \mathbb{Z}^+$,

$$RR(\mathcal{E}(k,5)) = \begin{cases} 4 & \text{for } k = 1,2,3 \\ 6 & \text{for } k = 4,13,14 \\ 7 & \text{for } k = 16,17,18,23 \\ 8 & \text{for } 5 \leq k \leq 12 \text{ and } k = 21 \\ 10 & \text{for } k = 19,24,26,27,28,29,33 \\ 11 & \text{for } k = 22,30,31,32,34,36,37,38,39,41,42,43,48 \\ 12 & \text{for } k = 15,35,44,46,47,53 \\ 13 & \text{for } k = 51,52 \\ 15 & \text{for } k = 20,25,40,45,49,50 \text{ and } k \geq 54. \end{cases}$$

Proof: Use the Maple package `FVR` with the following valid colorings (which are easily obtained):

n	valid colorings
4	$rrrb, rrbb, rbrb, rbbb$
5	$rrbbb, rbrbr, rbbbr$
6	$rrbbbr, rbrbrr, rbbbrr$
7	$rrbbbrr, rrbbbrb, rbrbrrr, rbrbrrb, rbbbrrr$
8	$rrbbbrrb, rbrbrrbr, rbbbrrrr$
9	$rrbbbrrbb, rbrbrrbrb, rbbbrrrrr.rbbbrrrrb$
10	$rrbbbrrbbr, rbbbrrrrbr, rbbbrrrrrr$
11	$rrbbbrrbbrr, rbbbrrrrrrr, rbbbrrrrbrr$
12	$rrbbbrrbbrrr, rbbbrrrrrrrr, rbbbrrrrbrrr$
13	$rrbbbrrbbrrrr, rrbbbrrbbrrrb$
14	$rrbbbrrbbrrrrr, rrbbbrrbbrrrrb, rrbbbrrbbrrrbr, rrbbbrrbbrrrbb$
15	none.

□

2.3 A Formula for $x + y + kz = 2w$

In [HS] and [GS] a formula for, in particular, $x+y+kz = w$ is given: $RR(x+y+kz = w) = (k+1)(k+4)+1$. In this section we provide a formula for the next important equation of this form, namely the one in this section's title. This appears to be the first formula given for a linear homogeneous equation $\mathcal{E}$ of more than three variables with a negative coefficient not equal to -1 (assuming, without loss of generality, at least as many positive coefficients as negative ones) that does not satisfy Rado's regularity condition.

Theorem 2.8: For $k \in \mathbb{Z}^+$,

$$RR(x + y + kz = 2w) = \begin{cases} \frac{k(k+4)}{4} + 1 & \text{if } k \equiv 0 \pmod 4 \\ \frac{(k+2)(k+3)}{4} + 1 & \text{if } k \equiv 1 \pmod 4 \\ \frac{(k+2)^2}{4} + 1 & \text{if } k \equiv 2 \pmod 4 \\ \frac{(k+1)(k+4)}{4} + 1 & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Proof: We begin with the lower bounds. Let N_i be one less than the stated formula for $k \equiv i \pmod 4$, with $i \in \{0, 1, 2, 3\}$. We will provide 2-colorings of $[1, N_i]$, for $i = 0, 1, 2, 3$, that admit no monochromatic solution to $x + y + kz = 2w$.

For $i = 0$, color all elements in $\left[1, \frac{k}{2}\right]$ red and all remaining elements blue. If we assume x, y, and z are all red, then $x + y + kz \geq k + 2$ so that for any solution we have $w \geq \frac{k}{2} + 1$. Thus, there is no red solution. If we assume x, y, and z are all blue, then $x+y+kz \geq \frac{k^2}{2}+2k+2 = 2\left(\frac{k(k+4)}{4}+1\right) > 2N_0$, showing that there is no blue solution.

For $i = 1$, color N_1 and all elements in $\left[1, \frac{k+1}{2}\right]$ red. Color the remaining elements blue. Similarly to the last case, we have no blue solution since $x + y + kz > 2(N_1 - 1)$. If we assume x, y, z and w are all red, then we cannot have all of x, y, z in $\left[1, \frac{k+1}{2}\right]$. If we do, since k is odd (so that we must have $x+y$ odd), then $x+y+kz \geq 1+2+k(1) = k+3 > 2\left(\frac{k+1}{2}\right)$. Thus, w must be blue. Now we assume, without loss of generality, that $x = N_1$. In this situation, we must have $w = N_1$. Hence, since $N_1 + y + kz = 2N_1$ we see that $y + kz = N_1$. Hence, $y, z \leq \frac{k+1}{2}$. But then $y + kz \leq \frac{k^2+2k+1}{2} < N_1$, a contradiction. Hence, there is no red solution under this coloring.

The cases $i = 2$ and $i = 3$ are similar to the above cases. As such, we provide the colorings and leave the details to the reader. For $i = 2$, we color N_2 and all elements in $\left[1, \frac{k}{2}\right]$ red, while the remaining elements are colored blue. For $i = 3$, color all elements in $\left[1, \frac{k+1}{2}\right]$ red and all remaining elements blue.

We now turn to the upper bounds. We let M_i be equal to the stated formula for $k \equiv i \pmod 4$, with $i \in \{0, 1, 2, 3\}$. We employ a "forcing" argument to determine the color of certain elements. We let R denote the set of red elements and B the set of blue elements. We denote by a 4-tuple (x, y, z, w) a solution to $x + y + kz = 2w$. In each of the following cases assume, for a contradiction, that there exists a 2-coloring of $[1, M_i]$ with no monochromatic solution to the equation. In each case we assume $1 \in R$.

`Case 1.` $k \equiv 0 \pmod 4$. We will first show that $2 \in R$. Assume, for a contradiction, that $2 \in B$. Then $2k + 2 \in R$ by considering $(2, 2k + 2, 2, 2k + 2)$. Also, $k + 1 \in B$ comes from the similar solution $(1, k + 1, 1, k + 1)$. Now, from $(3k + 3, 1, 1, 2k + 2)$ we have $3k + 3 \in B$. As a consequence, we see that $3 \in R$ by considering $(3, 3k + 3, 3, 3k + 3)$. From here we use $(3k + 1, 3, 1, 2k + 2)$ to see that $3k + 1 \in B$. But then $(3k + 1, k + 1, 2, 3k + 1)$ is a blue solution, a contradiction. Hence, $2 \in R$.

Now, since $1, 2 \in R$, in order for $\left(1, 1, 1, \frac{k}{2} + 1\right)$ not to be monochromatic, we have $\frac{k}{2} + 1 \in B$. Similarly, $\left(2, 2, 1, \frac{k}{2} + 2\right)$ gives $\frac{k}{2} + 2 \in B$. Consequently, so that $\left(\frac{k}{2} + 1, \frac{k}{2} + 1, \frac{k}{2} + 1, \frac{k^2}{4} + k + 1\right)$ is not monochromatic, we have $\frac{k^2}{4} + k + 1 \in R$.

Our next goal is to show that $\frac{k}{2} \in R$. So that $\left(\frac{k^2+2k}{4} + 1, \frac{k^2+2k}{4} + 1, 1, \frac{k^2}{4} + k + 1\right)$ is not red, we have $\frac{k^2+2k}{4} + 1 \in B$. In turn, to avoid $\left(\frac{k}{2} + 1, \frac{k}{2} + 1, \frac{k}{2}, \frac{k^2+2k}{4} + 1\right)$ being blue, we have $\frac{k}{2} \in R$, as desired.

So that $\left(\frac{k}{2}, \frac{k}{2}, 1, k\right)$ and $\left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k^2+2k}{4}\right)$ are not red, we have $k, \frac{k^2+2k}{4} \in B$. Using these in

$\left(k, k, \frac{k}{2}-1, \frac{k^2+2k}{4}\right)$ gives $\frac{k}{2}-1 \in R$. Since $\frac{k}{2}$ and $\frac{k}{2}-1$ are both red, $\left(\frac{k}{2}-1, \frac{k}{2}-1, \frac{k}{2}-1, \frac{k^2}{4}-1\right)$ gives us $\frac{k^2}{4}-1 \in B$ while $\left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}-1, \frac{k^2}{4}\right)$ gives us $\frac{k^2}{4} \in B$. This, in turn, gives us $\frac{k}{4} \in R$ by considering $\left(\frac{k^2}{4}, k, \frac{k}{4}, \frac{k^2+2k}{4}\right)$.

Now, from $\left(\frac{k^2+4k}{4}+1, 1, \frac{k}{4}+1, \frac{k^2+4k}{4}+1\right)$ we have $\frac{k}{4}+1 \in B$. We use this in the two solutions $\left(\frac{k^2}{4}-1, \frac{k}{2}+1, \frac{k}{4}+1, \frac{k^2+3k}{4}\right)$ and $\left(\frac{k^2}{4}, k, \frac{k}{4}+1, \frac{k^2+4k}{4}\right)$ to find that $\frac{k^2+3k}{4}, \frac{k^2+4k}{4} \in R$. But this gives us the red solution $\left(\frac{k^2+4k}{4}, \frac{k}{2}, \frac{k}{4}, \frac{k^2+3k}{4}\right)$, a contradiction.

`Case 2.` $k \equiv 1 \pmod 4$. The argument at the beginning of Case 1 holds for this case, so we have $2 \in R$. We consider two subcases.

`Subcase i.` $\frac{k+3}{4} \in R$. From $\left(\frac{k^2+4k+3}{4}, \frac{k+3}{4}, \frac{k+3}{4}, \frac{k^2+4k+3}{4}\right)$ we have $\frac{k^2+4k+3}{4} \in B$. This gives us $\frac{k+1}{2} \in R$ by considering $\left(k+1, \frac{k+1}{2}, \frac{k+1}{2}, \frac{k^2+4k+3}{4}\right)$ (where $k+1 \in B$ comes from $(1, 1, 2, k+1)$). We also have, from $(1, 2, 1, \frac{k+3}{2})$, that $\frac{k+3}{2} \in B$. Consequently, $\frac{k^2+5k+6}{4} \in R$ so that $\left(\frac{k+3}{2}, \frac{k+3}{2}, \frac{k+3}{2}, \frac{k^2+5k+6}{4}\right)$ is not monochromatic.

We next have that $\frac{k+5}{2} \in B$ so that $\left(3, 2, 1, \frac{k+5}{2}\right)$ is not monochromatic (we may assume that $k \geq 9$). Hence, $\frac{k^2+5k+10}{4} \in R$ by considering $\left(\frac{k+5}{2}, \frac{k+5}{2}, \frac{k+3}{2}, \frac{k^2+5k+10}{4}\right)$. But this gives us the monochromatic solution $\left(\frac{k^2+5k+10}{4}, \frac{k+1}{2}, \frac{k+3}{4}, \frac{k^2+5k+6}{4}\right)$, a contradiction.

`Subcase i.` $\frac{k+3}{4} \in B$. Via arguments similar to those in Subcase i, we have $\frac{k^2+4k+3}{4}, \frac{k^2+5k+6}{4} \in R$. From $\left(\frac{k^2+4k+3}{4}, \frac{k^2+2k+9}{4}, 1, \frac{k^2+5k+6}{4}\right)$ we have $\frac{k^2+2k+9}{4} \in B$. This gives us $\frac{k^2+15}{4} \in R$ by considering $\left(\frac{k^2+15}{4}, \frac{k+3}{4}, \frac{k+3}{4}, \frac{k^2+2k+9}{4}\right)$.

We now show that $\frac{k-1}{4} \in R$ by showing that for any $i \leq \frac{k-1}{4}$ we must have $i \in R$. To this end, assume, for a contradiction, that $i-1 \in R$ but $i \in B$ (where $i \geq 3$). From $(i, ik+i, i, ik+i)$ we have $ik+i \in R$. In turn we have $(i+1)k+i \in B$ by considering $(ik+i, ik+i, 2, (i+1)k+i)$. We next see from $((i+1)k+i, i, i+1, (i+1)k+i)$ that $i+1 \in R$. Using our assumption that $i-1 \in R$ in $(i-1, i+1, 2, k+i)$ we have $k+i \in B$. But then $((i+1)k+i, k+i, i, (i+1)k+i)$ is a blue solution, provided $(i+1)k+i \leq M_1$, which by the bound given on i is valid. By applying this argument to $i = 3, 4, \ldots, \frac{k-1}{4}$, in order, we see that all positive integers less than or equal to $\frac{k-1}{4}$ must be red. In particular, $\frac{k-1}{4} \in R$.

Using $\frac{k-1}{4} \in R$ in $\left(\frac{k^2+15}{4}, \frac{k+15}{4}, \frac{k-1}{4}, \frac{k^2+15}{4}\right)$ we have $\frac{k+15}{4} \in B$. This, in turn, gives us $\frac{k^2+4k+15}{4} \in R$ by considering $\left(\frac{k^2+4k+15}{4}, \frac{k+15}{4}, \frac{k+3}{4}, \frac{k^2+4k+15}{4}\right)$. For our contradiction, we see now that $\left(\frac{k^2+4k+15}{4}, \frac{k^2+15}{4}, 1, \frac{k^2+4k+15}{4}\right)$ is a red solution.

`Case 3.` $k \equiv 2 \pmod 4$. From Case 1 we have $\frac{k}{2}, \frac{k^2}{4}+k+1 \in R$ and $\frac{k}{2}+1, \frac{k^2}{4}-1, \frac{k^2+2k}{4} \in B$. From $\left(1, 1, \frac{k}{2}+2, \frac{k^2}{4}+k+1\right)$ we see that $\frac{k}{2}+2 \in B$. This gives us $\frac{k^2}{4}+k+2 \in R$ by considering $\left(\frac{k}{2}+2, \frac{k}{2}+2, \frac{k}{2}+1, \frac{k^2}{4}+k+2\right)$. Using this fact in $\left(\frac{k^2}{4}+k+2, \frac{k}{2}, \frac{k+2}{4}, \frac{k^2}{4}+k+1\right)$ we have $\frac{k+2}{4} \in B$. But then $\left(\frac{k^2}{4}-1, \frac{k}{2}+1, \frac{k+2}{4}, \frac{k^2+2k}{4}\right)$ is a blue solution, a contradiction.

`Case 4.` $k \equiv 3 \pmod 4$. Let $i \in R$. From Case 2 we may assume $i \geq 2$ so that $1, 2, \ldots, i$ are all red, By considering $(i, ik+i, i, ik+i)$ we have $ik+i \in B$ so that we may assume $k+1, 2k+2, \ldots, (i-1)k+(i-1), ik+i$ are all blue. Since $(i+1, (i-1)k+i-1, i+1, ik+i)$ is a solution, we have $i+1 \in R$. Hence, $i \in R$ for $1 \leq i \leq \frac{k+5}{4}$. In particular, $\frac{k+5}{4} \in R$. From Case 2 we also have $\frac{k+3}{2}, \frac{k+5}{2} \in B$. By considering $\left(\frac{k+3}{2}, \frac{k+5}{2}, \frac{k+3}{2}, \frac{k^2+5k+8}{4}\right)$ we have $\frac{k^2+5k+8}{4} \in R$. But then $\left(\frac{k^2+5k+8}{4}, 2, \frac{k+5}{4}, \frac{k^2+5k+8}{4}\right)$ is a red solution, a contradiction.

2.4 Concluding remarks on 2-color Rado numbers

The next important numbers to determine are in the first row of Table 1 (below). As such, it would be nice to have a formula for $RR(x+y+z=\ell w)$. One has not been discerned.

In analyzing Table 1 certain patterns emerge. The the following conjecture was put forth, but has since been discovered to be incorrect.

Conjecture: For $\ell \geq 2$ fixed and $k \geq \ell+2$, we have

$$RR(x+y+kz=\ell w) = \left(\left\lfloor \frac{k+\ell+1}{\ell} \right\rfloor\right)^2 + O\left(\frac{k}{\ell^2}\right),$$

where the "$O\left(\frac{k}{\ell^2}\right)$ part" depends on the residue class of k modulo ℓ^2.

Saracino and Wynne [SW] found the precise Rado numbers for the equation $x+y+kz=3w$, which relies on the residue of k mod 27 (not 9).

We end with a table of calculated values of $RR(\mathcal{E}(k,j))$ for small values of k and j. These were calculated by a standard backtrack algorithm. The program can be downloaded as `RADONUMBERS` at Aaron Robertson's website[2].

[2] `http://math.colgate.edu/~aaron/programs.html`

Table 1: Some Values of $RR(x+y+kz=\ell w)$

	$\ell=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$k=$ **1**	11	4	1	4	9	4	10	12	14	16	18	20	22	24	26	38	40	43	48	50	53	59	62
2	19	5	4	1	4	3	4	5	7	8	9	9	17	18	20	21	23	24	26	27	32	33	35
3	29	8	5	4	1	4	3	4	9	6	7	10	9	10	9	13	14	15	16	16	26	27	28
4	41	9	4	4	5	1	4	3	6	5	6	4	10	8	11	10	11	9	13	14	14	15	16
5	55	15	8	6	8	5	1	4	6	8	5	6	4	10	11	11	11	13	12	14	10	14	15
6	71	17	9	5	6	6	5	1	9	5	8	5	11	4	9	10	11	15	13	12	16	10	12
7	89	23	10	7	6	7	11	5	1	4	5	8	6	11	7	10	12	12	13	13	14	15	10
8	109	25	15	8	6	9	8	8	5	1	6	5	8	5	10	7	10	9	11	12	16	17	15
9	131	34	15	9	9	9	9	7	14	5	1	9	6	8	10	10	12	12	9	11	15	12	14
10	155	37	16	10	10	9	8	7	8	10	5	1	9	5	8	10	10	9	8	9	12	11	12
11	181	46	22	15	10	11	8	11	9	10	17	5	1	6	5	8	6	11	9	8	11	16	11
12	209	49	24	16	11	9	10	10	12	8	10	12	5	1	9	6	8	9	11	9	12	10	12
13	239	61	26	17	10	11	13	13	10	10	13	10	20	5	1	9	6	6	9	11	7	10	13
14	271	65	34	18	14	11	14	11	12	13	8	9	11	14	5	1	9	5	6	9	14	7	9
15	305	77	36	24	15	13	13	15	14	15	12	14	12	12	23	5	1	9	6	12	9	10	7
16	341	81	38	24	16	11	14	12	14	13	12	12	11	10	12	16	5	1	9	10	8	9	12
17	379	96	45	24	17	14	15	15	17	12	19	15	14	13	17	14	26	5	1	9	8	8	9
18	419	101	47	25	18	15	14	15	18	15	12	12	16	10	12	11	15	18	5	1	9	6	8
19	461	116	49	32	24	17	14	17	18	18	16	20	15	14	15	13	15	15	29	5	1	9	8
20	505	121	60	34	25	15	16	16	18	15	14	20	17	19	15	14	14	12	15	20	5	1	9
21	551	139	63	36	26	18	19	16	15	20	20	17	19	18	18	14	14	17	21	16	32	5	1
22	599	145	65	37	27	18	17	15	17	19	18	19	15	17	15	17	16	14	14	13	18	22	5
23	649	163	78	45	28	23	18	18	19	19	23	22	17	22	17	21	14	18	16	16	17	18	35
24	701	169	≥ 81	47	32	24	19	16	17	20	21	18	20	16	20	20	18	16	14	18	15	14	19
25	755	190	≥ 84	49	35	25	19	20	16	20	23	21	25	16	28	23	19	18	21	18	16	17	25
26	811	197	≥ 94	51	35	26	17	19	20	20	23	21	22	21	17	18	20	18	19	15	17	17	17
27	869	218	≥ 97	62	35	27	23	22	27	22	21	20	23	26	20	21	22	27	19	17	21	19	19
28	929	225	≥ 100	64	36	27	25	21	21	18	23	22	22	21	16	18	20	21	19	21	21	16	16
29	991	249	≥ 115	66	44	35	26	22	21	23	22	24	27	25	26	19	30	21	23	25	17	21	19
30	1055	257	≥ 118	68	46	36	24	19	23	22	22	21	25	24	30	17	20	24	22	21	22	18	16
31	1121	281	≥ 122	75	47	37	26	22	23	24	22	25	25	26	22	28	20	30	22	20	19	24	21
32	1189	289	≥ 138	77	49	38	27	24	24	23	25	24	26	24	27	24	18	20	21	22	24	25	19
33	1259	316	≥ 141	≥ 79	50	39	26	25	25	23	27	26	25	25	24	24	29	20	29	29	27	28	18
34	1331	325	≥ 147	≥ 81	60	40	34	24	23	24	26	23	27	28	29	27	28	20	21	26	23	26	24
35	1405	352	≥ 161	≥ 93	62	45	35	26	25	26	26	27	27	28	28	26	25	34	21	33	30	25	26
36	1481	361	≥ 165	≥ 96	63	45	36	24	27	26	27	30	27	24	30	27	29	27	20	22	27	22	27
37	1559	391	≥ 169	≥ 98	65	49	37	26	26	26	30	28	25	27	28	31	27	27	36	22	31	26	25
38	1639	401	≥ 188	≥ 100	65	48	38	28	28	25	26	27	30	30	29	30	31	32	38	22	23	28	24
39	1721	431	≥ 193	≥ 113	77	48	39	35	29	28	28	31	30	29	32	32	31	29	28	32	23	31	31
40	1805	441	≥ 196	≥ 116	79	49	39	35	29	25	28	26	31	27	31	32	31	32	32	30	22	24	29

	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$k=$ **1**	64	84	87	≥ 91	≥ 98	≥ 102	≥ 105	≥ 114	≥ 118	≥ 121	≥ 148	≥ 152	≥ 157	≥ 167	≥ 171	≥ 176	≥ 187
2	36	44	46	48	49	66	68	≥ 70	≥ 73	≥ 77	≥ 80	≥ 82	≥ 84	≥ 93	≥ 95	≥ 98	≥ 100
3	30	31	33	34	35	37	38	44	45	47	48	49	58	60	61	63	64
4	17	21	22	23	24	25	25	37	38	39	40	42	43	45	46	47	48
5	14	17	15	16	16	21	22	23	23	24	25	25	31	32	33	34	35
6	9	16	13	18	14	15	15	16	17	21	22	22	23	24	24	25	25
7	18	9	16	11	18	19	14	21	22	16	17	18	19	21	22	22	23
8	16	18	9	19	12	17	18	19	20	21	15	23	24	18	19	24	16
9	16	16	18	27	19	16	21	18	19	20	21	22	26	24	16	26	27
10	13	18	17	16	17	19	20	21	17	23	19	20	21	22	23	24	25
11	12	15	12	15	16	17	18	20	20	21	21	19	25	21	22	23	24
12	11	14	13	12	12	20	15	16	16	20	21	20	21	22	20	26	22
13	12	10	20	13	12	12	13	15	16	16	17	18	20	20	21	22	22
14	10	10	10	12	13	15	12	13	17	21	16	20	18	19	19	20	21
15	10	15	13	10	12	14	23	15	14	15	15	28	20	17	19	21	22
16	7	11	9	12	11	14	12	16	15	18	14	16	14	23	18	20	20
17	12	8	13	9	11	12	14	12	18	14	26	17	16	16	17	23	20
18	9	10	7	27	9	16	11	18	13	15	14	16	17	29	12	18	17
19	10	10	10	9	13	13	16	12	17	13	15	14	16	16	29	19	18
20	10	15	9	10	8	13	14	16	12	13	15	20	16	16	16	20	19
21	9	10	9	9	16	12	15	11	17	13	17	18	18	15	16	16	20
22	1	9	8	10	7	10	12	13	11	20	13	12	13	19	15	25	16
23	5	1	9	9	9	7	10	12	13	11	16	17	13	13	17	15	17
24	24	5	1	9	9	10	7	15	14	13	11	14	18	14	15	20	17
25	19	38	5	1	9	10	15	10	12	14	13	15	14	18	20	13	20
26	15	20	26	5	1	9	10	10	10	12	11	14	11	15	17	26	13
27	18	21	22	41	5	1	9	10	10	15	11	13	27	14	16	17	18
28	16	19	16	22	28	5	1	9	10	10	10	21	12	18	13	16	17
29	18	19	20	29	22	44	5	1	9	10	10	12	12	11	16	13	17
30	21	20	20	20	17	23	30	5	1	9	10	11	15	12	11	18	14
31	25	22	19	21	21	23	24	47	5	1	9	10	11	10	11	11	16
32	20	26	18	17	20	20	18	24	32	5	1	9	10	11	13	12	16
33	20	23	21	20	25	22	21	33	26	50	5	1	9	10	10	14	8
34	19	23	18	22	18	21	18	21	19	27	34	5	1	9	10	11	11
35	25	22	22	27	25	23	23	23	25	25	27	53	5	1	9	10	12
36	24	23	21	27	24	23	21	21	22	23	20	27	36	5	1	9	10
37	28	23	22	24	29	25	23	23	22	24	25	37	28	56	5	1	9
38	27	24	26	22	26	24	23	24	21	23	22	25	21	30	38	5	1
39	28	29	33	25	23	22	28	27	25	26	25	25	27	29	30	59	5
40	25	28	24	22	23	28	25	26	24	27	22	28	22	27	22	31	40

3. Off-diagonal 2-color Rado numbers

In [JS], the 2-color Rado numbers are determined for equations of the form $a_1x_1+\cdots+a_nx_n = z$ where one of the a_i's is 1. The case when $\min(a_1,\ldots,a_n) = 2$ is done in [HS], while the general case is settled in [GS].

In this section, we investigate the "off-diagonal" situation. To this end, for $r \in \mathbb{Z}^+$ define an off-diagonal Rado number for the equations $\mathcal{E}_i$, $0 \leq i \leq r-1$, to be the least integer N (if it exists) for which any r-coloring of $[1, N]$ must admit a monochromatic solution to $\mathcal{E}_i$ of color i for some $i \in [0, r-1]$.

Recall that the Ramsey number $R(k_1, k_2, \ldots, k_n)$ is the minimum integer that ensures the admittance of a monochromatic complete graph K_i of color i, meaning that they are themselves off-diagonal (unless $k_1 = k_2 = \cdots = k_n$, a case that has received particular attention).

In this section, when $r = 2$ we will prove the existence of such numbers and determine particular values and lower bounds in several specific cases when the two equations are of the form $a_1x_1 + \cdots + a_nx_n = z$. For convenience, we restate Rado's lesser-known theorem.

Theorem 3.1 [Rad]: Let $\mathcal{E}$ be a linear homogeneous equation with integer coefficients. Assume that $\mathcal{E}$ has at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits a monochromatic solution to $\mathcal{E}$.

3.1 Existence

Using Rado's lesser-known theorem, we offer the "off-diagonal" consequence.

Theorem 3.2: Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be linear homogeneous equations with integer coefficients. Assume that $\mathcal{E}_0$ and $\mathcal{E}_1$ each have at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits either a solution to $\mathcal{E}_0$ of the first color or a solution to $\mathcal{E}_1$ of the second color.

Proof: Let $a_0, a_1, b_0, b_1, c \in \mathbb{Z}^+$ and denote by $\mathcal{G}_i$ the equation $a_ix + b_iy = cz$ for $i = 0, 1$. Via the same argument given in the proof to Theorem 3.1, we may consider solutions to $\mathcal{G}_0$ and $\mathcal{G}_1$. (The coefficients on z may be taken to be the same in both equations by finding the lcm of the original coefficients on z and adjusting the other coefficients accordingly.)

Let the colors be red and blue. We want to show that any 2-coloring admits either a red solution to $\mathcal{G}_0$ or a blue solution to $\mathcal{G}_1$. From Theorem 3.1, we have monochromatic solutions to each of these equations. Hence, we assume, for a contradiction, that any monochromatic solution to $\mathcal{G}_0$ is blue and that any monochromatic solution to $\mathcal{G}_1$ is red. This gives us that for any $i \in \mathbb{Z}^+$, if ci is blue, then $(a_1 + b_1)i$ is red (else we have a blue solution to $\mathcal{G}_1$).

Now consider monochromatic solutions in $c\mathbb{Z}^+$. Via the obvious bijection between colorings of $c\mathbb{Z}^+$ and $\mathbb{Z}^+$ and the fact that linear homogeneous equations are unaffected by dilation, Theorem 0.1 gives us the existence of monochromatic solutions in $c\mathbb{Z}^+$. If cx, cy, cz solve $\mathcal{G}_0$ and are the same color, then they must be blue. Hence, $\hat{x} = (a_1 + b_1)x, \hat{y} = (a_1 + b_1)y$, and $\hat{z} = (a_1 + b_1)z$ are all red. But, $\hat{x}, \hat{y}, \hat{z}$ solve $\mathcal{G}_0$. Thus, we have a red solution to $\mathcal{G}_0$, a contradiction. □

3.2 Two Lower Bounds

Given the results in the previous section, we make a definition, which uses the following notation.

Notation: For $n \in \mathbb{Z}^+$ and $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, denote by $\mathcal{E}_n(\vec{a})$ the linear homogeneous equation $\sum_{i=1}^{n} a_i x_i = 0$.

Definition: For $k, \ell \geq 3, \vec{b} \in \mathbb{Z}^k$, and $\vec{c} \in \mathbb{Z}^\ell$, we let $RR(\mathcal{E}_k(\vec{b}), \mathcal{E}_\ell(\vec{c}))$ be the minimum integer N, if it exists, such that any 2-coloring of $[1, N]$ admits either a solution to $\mathcal{E}_k(\vec{b})$ of the first color or a solution to $\mathcal{E}_\ell(\vec{c})$ of the second color.

We now develop a general lower bound for certain types of those numbers guaranteed to exist by Theorem 1.1.

Theorem 3.3: For $k, \ell \geq 2$, let $b_1, b_2, \ldots, b_{k-1}, c_1, c_2, \ldots, c_{\ell-1} \in \mathbb{Z}^+$. Consider $\mathcal{E}_k = \mathcal{E}_k(b_1, b_2, \ldots, b_{k-1}, -1)$ and $\mathcal{E}_\ell = \mathcal{E}_\ell(c_1, c_2, \ldots, c_{\ell-1}, -1)$, written so that $b_1 = \min(b_1, b_2, \ldots, b_{k-1})$ and $c_1 = \min(c_1, c_2, \ldots, c_{\ell-1})$. Assume that $t = b_1 = c_1$. Let $q = \sum_{i=2}^{k-1} b_i$ and $s = \sum_{i=2}^{\ell-1} c_i$. Let (without loss of generality) $q \geq s$. Then

$$RR(\mathcal{E}_k, \mathcal{E}_\ell) \geq t(t + q)(t + s) + s.$$

For a concrete example, consider the equations:

$$2x + 4y + 7z + 11v = w \quad (\mathcal{E})$$

$$2x + 4y + 8z + 12v = w \quad (\mathcal{F})$$

Then we have $t = 2$, $s = 22$, $q = 24$, and thus $RR(\mathcal{E}, \mathcal{F}) \geq 1270$.

Proof: Let $N = t(t+q)(t+s)+s$ and consider the 2-coloring of $[1, N-1]$ defined by coloring $[s+t, (q+t)(s+t)-1]$ red and its complement blue. We will show that this coloring avoids red solutions to $\mathcal{E}_k$ and blue solutions to $\mathcal{E}_\ell$.

We first consider any possible red solution to $\mathcal{E}_k$. The value of x_k would have to be at least $t(s+t)+q(s+t)=(q+t)(s+t)$. Thus, there is no suitable red solution. Next, we consider $\mathcal{E}_\ell$. If $\{x_1,x_2,\ldots,x_{\ell-1}\}\subseteq[1,s+t-1]$, then $x_\ell<(q+t)(s+t)$. Hence, the smallest possible blue solution to $\mathcal{E}_\ell$ has $x_i\in[(q+t)(s+t),N-1]$ for some $i\in[1,\ell-1]$. However, this gives $x_\ell\geq t(q+t)(s+t)+s>N-1$. Thus, there is no suitable blue solution. □

The case when $k=\ell=2$ in Theorem 3.3 can be improved somewhat in certain cases, depending upon the relationship between t, q, and s. This result is presented below.

Theorem 3.4: Let $t,j\in\mathbb{Z}^+$. Let $\mathcal{F}_j^t$ represent the equation $tx+jy=z$. Let $q,s\in\mathbb{Z}^+$ with $q\geq s\geq t$. Define $m=\frac{\gcd(t,q)}{\gcd(t,q,s)}$. Then

$$RR(\mathcal{F}_q^t,\mathcal{F}_s^t)\geq t(t+q)(t+s)+ms.$$

Proof: Let $N=t(t+q)(t+s)+ms$ and consider the 2-coloring χ of $[1,N-1]$ defined by coloring

$$R=[s+t,(q+t)(s+t)-1]\cup\{t(t+q)(t+s)+is:1\leq i\leq m-1\}$$

red and $B=[1,N-1]\setminus R$ blue. We will show that this coloring avoids red solutions to $\mathcal{F}_q^t$ and blue solutions to $\mathcal{F}_s^t$.

We first consider any possible red solution to $\mathcal{F}_q^t$. The value of z would have to be at least $t(s+t)+q(s+t)=(q+t)(s+t)$ and congruent to 0 modulo m. Since $t(t+q)(t+s)\equiv 0 \pmod{m}$ but $is\not\equiv 0 \pmod{m}$ for $1\leq i\leq m-1$, there is no suitable red solution. Next, we consider $\mathcal{F}_s^t$. If $\{x,y\}\subseteq[1,s+t-1]$, then $s+t\leq z<(q+t)(s+t)$. Hence, the smallest possible blue solution to $\mathcal{F}_s^t$ has x or y in $[(q+t)(s+t),N-1]$. However, this gives $z\geq t(q+t)(s+t)+s>N-1$. By the definition of the coloring, z must be red. Thus, there is no suitable blue solution to $\mathcal{F}_s^t$. □

3.3 Some Exact Numbers

In this section, we will determine some of the values of $RR_1(q,s)=RR(x+qy=z,x+sy=z)$, where $1\leq s\leq q$. The subscript 1 is present to emphasize the fact that we are using $t=1$ as defined in Theorem 3.4. In this section we will let $RR_t(q,s)=RR(tx+qy=z,tx+sy=z)$ and we will denote the equation $tx+jy=z$ by $\mathcal{F}_j^t$.

Theorem 3.5: Let $1\leq s\leq q$. Then

$$RR_1(q,s)=\begin{cases}2q+2\left\lfloor\frac{q+1}{2}\right\rfloor+1 & \text{for } s=1\\ (q+1)(s+1)+s & \text{for } s\geq 2.\end{cases}$$

Proof: We start with the case $s = 1$. Let $N = 2q + 2\left\lfloor \frac{q+1}{2} \right\rfloor + 1$. We first improve the lower bound given by Theorem 3.3 for this case.

Let γ be the 2-coloring of $[1, N-1]$ defined as follows. The first $2\lfloor \frac{q+1}{2} \rfloor - 1$ integers alternate colors with the color of 1 being blue. We then color $\left[2\lfloor \frac{q+1}{2} \rfloor, 2q+1\right]$ red. We color the last $2\lfloor \frac{q+1}{2} \rfloor - 1$ integers with alternating colors, where the color of $2q+2$ is blue.

First consider possible blue solutions to $x + y = z$. If $x, y \leq 2\lfloor \frac{q+1}{2} \rfloor - 1$, then $z \leq 2q$. Under γ, such a z must be red. Now, if exactly one of x and y is greater than $2q+1$, then z is odd and greater than $2q+1$. Again, such a z must be red. Finally, if both x and y are greater than $2q+1$, then z is too big. Hence, γ admits no blue solution to $x + y = z$.

Next, we consider possible red solutions to $x + qy = z$. If $x, y \leq \lfloor \frac{q+1}{2} \rfloor - 1$, then z must be even. Also, since x and y must both be at least 2 under γ, we see that $z \geq 2q+2$. Under γ, such a z must be blue. If one (or both) of x or y is greater than $\lfloor \frac{q+1}{2} \rfloor - 1$, then $z \geq N-1$, with equality possible. However, with equality, the color of z is blue. Hence, γ admits no red solution to $x + qy = z$.

We move onto the upper bound. Let χ be a 2-coloring of $[1, N]$ using the colors red and blue. Assume, for a contradiction, that there is no red solution to $\mathcal{F}_q^1$ and no blue solution to $\mathcal{F}_1^1$. We break the argument into 3 cases.

`Case 1.` 1 is red. Then $q+1$ must be blue since otherwise $(x, y, z) = (1, 1, q+1)$ would be a red solution to $\mathcal{F}_q^1$. Since $(q+1, q+1, 2q+2)$ satisfies $\mathcal{F}_1^1$, we have that $2q+2$ must be red. Now, since $(q+2, 1, 2q+2)$ satisfies $\mathcal{F}_q^1$, we see that $q+2$ must be blue. Since $(2, q+2, q+4)$ satisfies $\mathcal{F}_1^1$ we have that $q+4$ must be red. This implies that 4 must be blue since $(4, 1, q+4)$ satisfies $\mathcal{F}_q^1$. But then $(2, 2, 4)$ is a blue solution to $\mathcal{F}_1^1$, a contradiction.

`Case 2.` 1 is blue and q is odd. Note that in this case we have $N = 3q+2$. Since 1 is blue, 2 must be red, which, in turn, implies that $2q+2$ must be blue. Since $(q+1, q+1, 2q+2)$ solves $\mathcal{F}_1^1$, we see that $q+1$ must be red. Now, since $(j, 2q+2, 2q+j+2)$ solves $\mathcal{F}_1^1$ and $(j+2, 2, 2q+j+2)$ solves $\mathcal{F}_q^1$, we have that for any $j \in \{1, 3, 5, \ldots, q\}$, the color of j is blue. With 2 and q both red, we have that $3q$ is blue, which implies that $3q+1$ must be red. Since $(q+1, 2, 3q+1)$ solves $\mathcal{F}_q^1$, we see that $q+1$ must be blue, and hence $q+2$ is red. Considering $(q+2, 2, 3q+2)$, which solves $\mathcal{F}_q^1$, and $(q, 2q+2, 3q+2)$, which solves $\mathcal{F}_1^1$, we have an undesired monochromatic solution, a contradiction.

`Case 3.` 1 is blue and q is even. Note that in this case we have $N = 3q+1$. As in Case 2, we argue that for any $j \in \{1, 3, 5, \ldots, q-1\}$, the color of j is blue. As in Case 2, both 2 and $q+1$ must be red, so that $3q+1$ must be blue. But $(q-1, 2q+2, 3q+1)$ is then a blue solution to $\mathcal{F}_1^1$, a contradiction.

Next, consider the cases when $s \geq 2$. From Theorem 3.3, we have $RR_1(q, s) \geq (q+1)(s+$

$1) + s$. We proceed by showing that $RR_1(q,s) \leq (q+1)(s+1)+s$.

In the case when $s = 1$ we used an obvious "forcing" argument. As such, we have automated the process in the Maple package SCHAAL, available for download from Aaron Robertson's website[3]. The package is detailed in the next subsection, but first we finish the proof. Using SCHAAL we find the following (where we use the fact that $s \geq 2$):

1) If 1 is red, then the elements in $\{s, q+s+1, qs+q+s+1\}$ must be both red and blue, a contradiction.

2) If 1 is blue and $s-1$ is red, then the elements in $\{1, 2, 2q-1, 2s+1, 2q+1, 2q+2s-1, 2q+2s+1\}$ must be both red and blue, a contradiction.

3) If 1 and $s-1$ are both blue, the analysis is a bit more involved. First, by assuming $s \geq 2$ we find that 2 must be red and s must be blue. Hence, we cannot have $s = 2$ or $s = 3$, since if $s = 2$ then 2 is both red and blue, and if $s = 3$ then since $s-1$ is blue, we again have that 2 is both red and blue. Thus, we may assume that $s \geq 4$. Using SCHAAL with $s \geq 4$ now produces the result that the elements in $\{4, s+1, q+1, 2s-1, 2s, q+2s+1, 3s+1, 5q+1, 4q+s+1, 4q+2s-1, 4q+2s, 4q+3s+1, 5q+2s+1, qs-3q+1, qs-3q+2s+1, qs-3q+s-1, qs+q+1, qs+q+s-1, qs+q+2s+1\}$ must be both red and blue, a contradiction.

This completes the proof of the theorem. □

Using the above theorem, we offer the following corollary.

Corollary 3.6: For $k, \ell \in \mathbb{Z}^+$, let $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in \mathbb{Z}^+$. Assume $\sum_{i=1}^k a_i \geq \sum_{i=1}^\ell b_i$. Then

$$RR_1\left(x+\sum_{i=1}^k a_i y_i = z, x+\sum_{i=1}^\ell b_i y_i = z\right) = \begin{cases} 2\sum_{i=1}^k a_i + 2\left\lfloor \dfrac{\sum_{i=1}^k a_i + 1}{2} \right\rfloor + 1 & \text{for } \sum_{i=1}^\ell b_i = 1 \\ \left(\sum_{i=1}^k a_i + 1\right)\left(\sum_{i=1}^\ell b_i + 1\right) + \sum_{i=1}^\ell b_i & \text{for } \sum_{i=1}^\ell b_i \geq 2. \end{cases}$$

Proof: We start by proving that the coloring given in the proof of Theorem 3.5 which provides the lower bound for the case $s = 1$ also provides (with a slight modification) a lower bound for the case when $\sum_{i=1}^\ell b_i = 1$. In this situation, we must show that the coloring where the first $2\lfloor \frac{\sum_{i=1}^k a_i + 1}{2} \rfloor - 1$ integers alternate colors with the color of 1 being blue. We then color $\left[2\lfloor \frac{\sum_{i=1}^k a_i + 1}{2} \rfloor, 2\sum_{i=1}^k a_i + 1\right]$ red. We color the last $2\lfloor \frac{\sum_{i=1}^k a_i + 1}{2} \rfloor - 1$ integers with alternating colors, where the color of $2\sum_{i=1}^k a_i + 2$ is blue. An obvious parity argument shows that there is no blue solution to $x + y = z$ (this is the case when $\sum_{i=1}^\ell b_i = 1$) exists, so it remains

[3] `http://math.colgate.edu/~aaron`

to show that no red solution to $x + \sum_{i=1}^{k} a_i y_i = z$ exists under this coloring. Now, if x and all the y_i's are less than $2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor$, then z would be even and have value at least $2\sum_{i=1}^{k} a_i + 2$. This is not possible, so at least one of $x, y_1, \ldots, y_k$ must have value at least $2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor$. If $x \geq 2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor$, then $z \geq 2\sum_{i=1}^{k} a_i + 2\left\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \right\rfloor$. Hence, either z is blue or too big. So, assume, without loss of generality, that $y_1 \geq 2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor$. If $a_1 = 1$, then $z = x+y_1+\sum_{i=2}^{k} a_i y_i \geq 2+2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor+2\sum_{i=2}^{k} a_i = 2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor+2\sum_{i=1}^{k} a_i$ and again either z is blue or too big. If $a_1 \geq 2$ (and we may assume that $k \geq 2$ so that $\sum_{i=1}^{k} a_i + 1 \geq 4$), then $z = x + a_1 y_1 + \sum_{i=2}^{k} a_i y_i > a_1 \cdot 2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor + 2\sum_{i=2}^{k} a_i y_i \geq 2(a_1 + \lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor) + 2\sum_{i=2}^{k} a_i y_i = 2\lfloor \frac{\sum_{i=1}^{k} a_i+1}{2} \rfloor) + 2\sum_{i=1}^{k} a_i y_i$ and z is too big.

Next, by coupling the above lower bound with Theorem 2.1 (using $t = 1$), it remains to prove that the righthand sides of the theorem's equations serve as upper bounds for $N = RR_1(x+\sum_{i=1}^{k} a_i y_i = z, x+\sum_{i=1}^{\ell} b_i y_i = z)$. Letting $q = \sum_{i=1}^{k} a_i$ and $s = \sum_{i=1}^{\ell} b_i$, any solution to $x + qy = z$ (resp., $x + sy = z$) is a solution to $x + \sum_{i=1}^{k} a_i y_i$ (resp., $x + \sum_{i=1}^{\ell} b_i y_i = z$) by letting all y_i's equal y. Hence, $N \leq RR_1(q, s)$ and we are done. $\square$

Remark. When $a_i = 1$ for $1 \leq i \leq k$, $\ell = 1$, and $b_1 = 1$ the numbers in Corollary 3.2 are called the off-diagonal generalized Schur numbers. In this case, the values of the numbers have been determined [RS].

3.4 About the Maple Package SCHAAL

This package is used to try to automatically provide an upper bound for the off-diagonal Rado-type numbers $RR_t(q, s)$. The package employs a set of rules to follow, while the overall approach is an implementation of the above "forcing" argument.

Let $t \geq 2$ be given, keep $q \geq s$ as parameters, and define $N = tqs + t^2 q + (t^2 + 1)s + t^3$. We let $\mathcal{R}$ and $\mathcal{B}$ be the set of red, respectively blue, elements in $[1, N]$. The package SCHAAL uses the following rules.

For $x, y \in \mathcal{R}$,

R1) if $q|(y - tx)$ and $y - tx > 0$, then $\frac{y-tx}{q} \in \mathcal{B}$;

R2) if $t|(y - qx)$ and $y - qx > 0$, then $\frac{y-qx}{t} \in \mathcal{B}$;

R3) if $(q + t)|x$ then $\frac{x}{q+t} \in \mathcal{B}$.

For $x, y \in \mathcal{B}$,

B1) if $s|(y-tx)$ and $y-tx>0$, then $\frac{y-tx}{s}\in\mathcal{R}$;

B2) if $t|(y-sx)$ and $y-sx>0$, then $\frac{y-sx}{t}\in\mathcal{R}$;

B3) if $(s+t)|x$ then $\frac{x}{s+t}\in\mathcal{R}$.

We must, of course, make sure that the elements whose colors are implied by the above rules are in $[1,N]$. This is done by making sure that the coefficients of qs, q, and s, as well as the constant term are nonnegative and at most equal to the corresponding coefficients in $tqs+t^2q+(t^2+1)s+t^3$ (hence the need for t to be an integer and not a parameter). See the Maple code for more details.

The main program of `SCHAAL` is `dan`. The program `dan` runs until $\mathcal{R}\cap\mathcal{B}\neq\emptyset$ or until none of the above rules produce a color for a new element.

3.5 Some Diagonal Results Using `SCHAAL`

Included in the package `SCHAAL` is the program `diagdan`, which is a cleaned-up version of `dan` in the case when $q=s$. Using `diagdan` we are able to reprove the main results found in [HS] and [JS]. However, our program is not designed to reproduce the results in [GS], which keeps t as a parameter and confirms the conjecture of Hopkins and Schaal [HS] that $R_t(q,q)=tq^2+(2t^2+1)q+t^3$.

Theorem 3.7: (*Jones and Schaal [JS]*) $R_1(q,q)=q^2+3q+1$

Proof: By running `diagdan`$(\{1\},\{\},1,q)$ we find immediately that the elements in $\{1,2,q,2q+1,q^2+2q+1\}$ must be both red and blue, a contradiction. □

Theorem 3.8: (*Hopkins and Schaal [HS]*) $R_2(q,q)=2q^2+9q+8$

Proof: By running `diagdan`$(\{1\},\{q\},2,q)$ we find immediately that the elements in $\{q+2,2q^2+5q,\frac{1}{2}(q^2+3q)\}$ must be both red and blue. We then run `diagdan`$(\{1,q\},\{\},2,q)$ and find that the elements in $\{2,q+2,2q,6q,q^2+6q\}$ must be both red and blue. The program ran for about 10 seconds to obtain this proof. □

3.6 Some Values of $RR_t(q,s)$

This section concludes with values of $RR_t(q,s)$ for small values of t, q and s.

t	q	s	Value		t	q	s	Value
2	3	2	43		3	5	4	172
2	4	2	50		3	6	4	201
2	5	2	58		3	7	4	214
2	6	2	66		3	8	4	235
2	7	2	74		3	9	4	264
2	8	2	82		3	10	4	277
2	9	2	90		3	6	5	231
2	10	2	98		3	7	5	245
2	4	3	66		3	8	5	269
2	5	3	73		3	9	5	303
2	6	3	86		3	10	5	317
2	7	3	93		3	7	6	276
2	8	3	106		3	8	6	303
2	9	3	112		3	9	6	330
2	10	3	126		3	10	6	357
2	5	4	88		3	8	7	337
2	6	4	100		3	9	7	381
2	7	4	112		3	10	7	397
2	8	4	124		3	9	8	420
2	9	4	136		3	10	8	437
2	10	4	148		3	10	9	477
2	6	5	122		4	5	4	292
2	7	5	131		4	6	4	324
2	8	5	150		4	7	4	356
2	9	5	159		4	8	4	388
2	10	5	178		4	9	4	432*
2	7	6	150		4	10	4	452
2	8	6	166		4	6	5	370
2	9	6	182		4	7	5	401
2	10	6	198		4	8	5	452
2	8	7	194		4	9	5	473
2	9	7	205		4	10	5	514
2	10	7	230		4	7	6	446
2	9	8	228		4	8	6	492
2	10	8	248		4	9	6	526
2	10	9	282		4	10	6	566
3	4	3	129		4	8	7	556
3	5	3	147		4	9	7	579
3	6	3	165		4	10	7	630
3	7	3	192*		4	9	8	632
3	8	3	201		4	10	8	680
3	9	3	219		4	10	9	746
3	10	3	237		5	11	5	820*

Table 2: Small Values of $RR_t(q,s)$

These values were calculated by matching Theorem 3.4's lower bound with the Maple package SCHAAL's upper bound. We use SCHAAL by letting 1 be red and then letting 1 be blue. In many cases this is sufficient, however in many of the remaining cases, we must consider subcases depending upon whether 2 is red or blue. If this is still not sufficient, we consider subsubcases depending upon whether the value in the table, the integer 3, the integer 4, or the integer 5, is red or blue. This is sufficient for all values in Table 2, expect for those marked with an *. This is because, except for those three values marked with an *, all values agree with the lower bound given by Theorem 3.4. For these three exceptional values, we can increase the lower bound given in Theorem 3.4.

Theorem 3.9: Let $t \geq 3$. Then $R_t(2t+1,t) \geq 6t^3+2t^2+4t$.

Proof: It is easy to check that the 2-coloring of $[1, 6t^3+2t^2+4t-1]$ defined by coloring $\{1,2,6t\} \cup \{6t+3,\dots,6t^2+2t-1\} \cup \{6t^2+2t \leq i \leq 12t^2+4t : i \equiv 0 \pmod{t}\}$ red and its complement blue avoids red solutions to $tx+(2t+1)y=z$ and blue solutions to $tx+ty=z$. (We use $t>2$ so that $6t$ is the minimal red element that is congruent to 0 modulo t.) □

Remark: The lower bound in Theorem 3.9 is not tight. For example, when $t=6$, the 2-coloring of $[1,1392]$ given by coloring $\{1,2,3,37,39,40,41,43,46,47,48,49,50,52,56\} \cup [58,228] \cup \{234 \leq i \leq 558 : i \equiv 0 \pmod{6}\} \cup \{570,576,594,606,612,648,684\}$ red and its complement blue avoids red solutions to $6x+13y=z$ and blue solutions to $6x+6y=z$. Hence, $RR_t(2t+1,t) > 6t^3+2t^2+4t$ for $t=6$.

We are unable to explain why $(b,c)=(2t+1,t)$ produces these "anomolous" values while others, e.g., $(b,c)=(2t-1,t)$, appear not to do so.

4. Open problems and concluding remarks

There are a great many open questions in Ramsey theory, some of which stem from this work specifically. Open questions stemming from our research include:

- What are the precise Rado numbers $RR(x+y+kz=\ell w)$? What relationships, if any, are there between these numbers for various k, ℓ? In particular, what are the Rado numbers when $\ell = 4$? Do they depend on k modulo 16, 64, or something else?
- What are the precise Rado numbers for different or more general equations? Of particular interest are those with more than one negative coefficient, or with a negative coefficient besides 1 or 2. A promising example would be the Sidon equation $x+y=w+z$, whose solutions have the helpful property of being translation-invariant.
- What are the precise off-diagonal Rado numbers of the form $RR(tx+qy=z, tx+sy=z)$? Why is the lower bound given in theorem 3.4 apparently correct except in the case $q=2t+1$, $s=t$?

The scope of existing research is almost exclusively on linear equations, mostly homogeneous as well. Open questions looking beyond this scope include:

- Does Rado's lesser-known theorem have a nonhomogeneous counterpart either? What necessary and/or sufficient conditions might there be for the 2-regularity of $c_1x_1+\cdots+c_nx_n=b$? This question has been investigated in [JS2].
- Does Rado's lesser-known theorem have a version for 3-regularity. How strict are the conditions on an equation being 3-regular? This question has been investigated in [RTS].
- Is the equation $x^2+y^2=z^2$ regular, or at least 2-regular? This question is proposed by Erdős, as relayed by Graham [Gr].
- What other nonlinear diophantine equations can be characterized as regular? What conditions can we impose on such regularity?
- What can be said of Rado-type results over different additive groups (e.g. $\mathbb{Z}/n\mathbb{Z}$)?

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