15.3: Double Integrals over General Regions

Marius Ionescu

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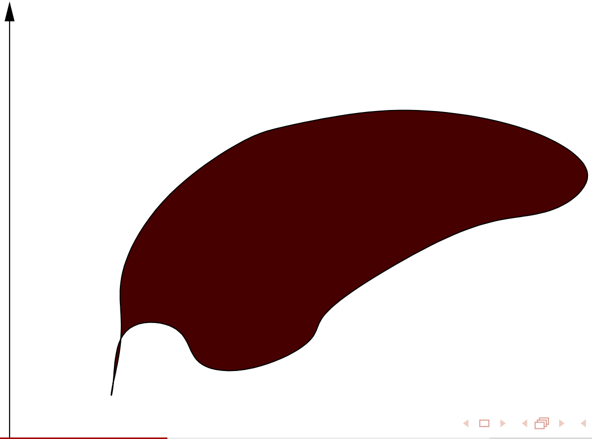
Double Integrals over General Regions

Fact

We want to integrate a function f over bounded regions D of more general shape:



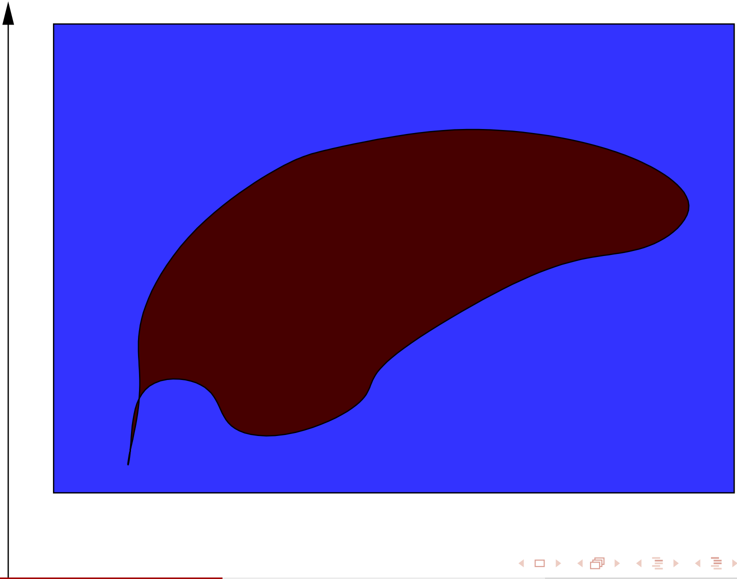
First things first: Definition

Definition

If D is a bounded region, then we define a new function F with domain a rectangle R that contains D by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{otherwise} \end{cases}$$

Definition



Definition

Definitions

The **double integral of** f **over** D is

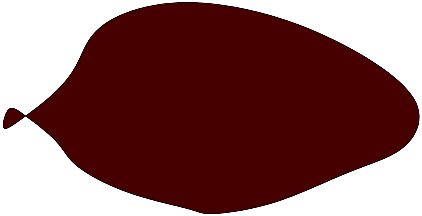
$$\iint_D f(x,y)dA = \iint_R F(x,y)dA.$$

Domains of type I

Definition

A domain D is of type I if it lies between the graphs of two continuous functions of x:

$$D = \{(x, y) \mid a \leq x \leq b,\, g_1(x) \leq y \leq g_2(x)\}$$



Double Integrals over Domains of type I

Fact

If D is a region of type I and f is continuous then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Examples

Evaluate the following double integrals:

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- $\iint_D \frac{y}{x^5+1} dA$, where $D = \{(x, y) \,|\, 0 \leq 1, 0 \leq y \leq x^2\}$

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- $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

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- $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.
- $\iint_D (x^2 + 2y) dA$, where D is bounded by $y = x$, $y = x^2$,$x \geq 0$.

Domains of type II

Definition

A domain D is of type II if it can be expressed as

$$D = \{(x, y) : c \leq y \leq d,\, h_1(y) \leq x \leq h_2(y)\}.$$

Double Integrals over Domains of type II

Fact

If D is a region of type II and f is continuous then

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Examples

Examples

Evaluate the following integrals

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- $\iint_D xy dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Examples

Examples

Evaluate the following integrals

- $\iint_D xydA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
- $\iint_D y^2 e^{xy} dA$, where D is the region bounded by $y = x$, $y = 4$, $x = 0$.

More Examples

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- Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)dydx$.

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- Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)dydx$.
- Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

More Examples

Examples

- Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) dy dx$.
- Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.
- Find the volume of the solid under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

Properties of the double integral

Fact

Properties of the double integral:

- $\iint_D [f(x,y) + g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA.$
- $\iint_D cf(x,y)dA = c\iint_D f(x,y)dA.$
- If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then $\iint_D f(x,y)dA \geq \iint_D g(x,y)dA.$
- If $D = D_1 \bigcup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then $\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA.$

Properties of the double integral (cont'd)

Fact

Properties of the double integral:

- $\iint_D 1 dA = \text{area of } D = A(D)$.
- *If* $m \le f(x,y) \le M$, *then* $mA(D) \le \iint_D f(x,y) dA \le MA(D)$.

Examples

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- Find the are of the triangle with vertices $(0, 0)$, $(5, 0)$, and $(5, 4)$ (using double integrals).

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- Find the are of the triangle with vertices $(0, 0)$, $(5, 0)$, and $(5, 4)$ (using double integrals).
- Estimate the integral $\iint_D e^{\sin x \cos y} dA$, where D is the disk with center the origin and radius 2.