Changing the index of an infinite series:

Suppose you have
\[ \sum_{n=1}^{\infty} (2n+1) x^{n-1} \]
and you would rather start with \( n = 0 \). You can shift your indexes as follows. Let \( n - 1 = k \) so that when \( n = 1 \), \( k = 0 \). (Notice that we choose this since under the \( \sum \) is \( n = 1 \) and we want \( n - 1 = 0 \).) Solve for \( n = k + 1 \), and we replace every \( n \) with \( k + 1 \) in our infinite series:
\[ \sum_{k=0}^{\infty} (2(k+1) + 1) x^{(k+1)-1} \Rightarrow \sum_{k=0}^{\infty} (2k + 3) x^k \]
Since both \( k \) and \( n \) are just indexing variables, we can reuse \( n \) in the above series on the right to get
\[ \sum_{n=0}^{\infty} (2n + 3) x^n \]
Notice that
\[ \sum_{n=1}^{\infty} (2n+1) x^{n-1} = 3 + 5x + 7x^2 + 9x^3 + \cdots \]
\[ \sum_{n=0}^{\infty} (2n + 3) x^n = 3 + 5x + 7x^2 + 9x^3 + \cdots \]

Suppose you have
\[ \sum_{n=0}^{\infty} (-1)^n x^{n+1}, \]
but you need the exponent of \( x \) to be \( n \) instead of \( n + 1 \). Then you can let \( k = n + 1 \) and shift the index. Here, \( n = k - 1 \) and
\[ \sum_{k-1=0}^{\infty} (-1)^{k-1} x^{(k-1)+1} \Rightarrow \sum_{k=1}^{\infty} (-1)^{k-1} x^k \]
Reusing the letter \( n \), we have
\[ \sum_{n=1}^{\infty} (-1)^{n-1} x^n \]
Checking our series,
\[ \sum_{n=0}^{\infty} (-1)^n x^{n+1} = x - x^2 + x^3 - x^4 + \cdots \]
\[ \sum_{n=1}^{\infty} (-1)^{n-1} x^n = x - x^2 + x^3 - x^4 + \cdots \]