Math 213 – Calculus III **Exam 1** **Spring 2002**

Name: SOLUTIONS

1. Match each function to one of the given graphs, and to one of the given contour diagrams. The plots are given in a separate handout.

$f(x,y)$	Graph	Contour Diagram
$x^2 + \left(\frac{3y}{2}\right)^2$	A	2
$x^2 - y^2$	G	8
$\frac{x}{2} + y$	H	1
$-x + y$	D	3
$\sin(\pi(x-y))$	B	5
xy	C	6
$\sqrt{x^2 + \left(\frac{3y}{2}\right)^2}$	F	7
$x + 2y^2$	E	4

2. (a) Find the equation of the plane that contains the points $(1,0,0)$, $(0,-3,0)$, and $(0,0,4)$.

First we'll find the slopes in the x and y directions.
The points $(1,0,0)$ and $(0,0,4)$ are in the xz plane, so $m = \frac{\Delta z}{\Delta x} = \frac{4-0}{0-1} = -4$.
The points $(0,3,0)$ and $(0,0,4)$ are in the yz plane, so $n = \frac{\Delta z}{\Delta y} = \frac{4-0}{0-(-3)} = \frac{4}{3}$.
Then, using $x_0 = 0$, $y_0 = 0$ and $z_0 = 4$, we have

$$z = 4 - 4x + \frac{4}{3}y.$$

(b) Find the equation of a sphere centered at $(1,2,3)$ that is tangent to the yz-plane.

The distance from $(1,2,3)$ to the yz plane is 1, so we want a sphere centered at $(1,2,3)$ with radius 1:

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1.$$

3. (a) Consider the function

$$f(x, y) = y^2 - \sin(x - y) - 3.$$

Find a function $g(x, y, z)$ such that the graph of f is the level surface $g(x, y, z) = 2$. The graph is given by the set of points where $z = y^2 - \sin(x - y) - 3$, which we can rewrite as $y^2 - \sin(x - y) - 3 - z = 0$, or $y^2 - \sin(x - y) - z - 1 = 2$, so one possible answer is

$$g(x, y, z) = y^2 - \sin(x - y) - z - 1.$$

(b) For each of the following functions, determine if the level surface $g(x, y, z) = 1$ can be expressed as the graph of a function $f(x, y)$. If it is not possible, explain why not. If is it possible, find the function $f(x, y)$.

i. $g(x, y, z) = x^2 + 2y^2 - z^2$ NO.
We have $x^2 + 2y^2 - z^2 = 1$, and if we try to solve for z, we find $z = \pm\sqrt{x^2 + 2y^2 - 1}$. For each point (x, y) there are two possible values of z (at least where $x^2 + 2y^2 - 1 > 0$), so we can not express this set of points as the graph of a function $f(x, y)$.

ii. $g(x, y, z) = -x - y + 3z + 8$ YES.
In this case $-x - y + 3z + 8 = 1$ can be solved for z, and we find $z = (x + y - 7)/3$; therefore $f(x, y) = (x + y - 7)/3$.

iii. $g(x, y, z) = e^{x+y+z}$ YES.
To solve $e^{x+y+z} = 1$ for z, take the $\ln$ of both sides to get $x + y + z = \ln 1 = 0$, then $z = -x - y$, so $f(x, y) = -x - y$.

4. For each of the following, find the indicated partial derivative.

(a) $f(x, y) = xy - 3\sin x$, $\quad f_x(x, y) = y - 3\cos x$

(b) $g(x, y) = e^{x-y}$, $\quad f_y(x, y) = e^{x-y}(-1) = -e^{x-y}$

(c) $h(s, t) = \sqrt{s^2 + st + t^2}$, $\quad \dfrac{\partial h}{\partial s} = \dfrac{1}{2}\left(s^2 + st + t^2\right)^{-1/2}(2s + t)$

(d) $f(x, y) = \ln(x^2 + 2y)$, $\quad f_y(x, y) = \dfrac{2}{x^2 + 2y}$

(e) $P = bL^\alpha K^{1-\alpha}$, $\quad \dfrac{\partial P}{\partial K} = (1 - \alpha)bL^\alpha K^{-\alpha}$

5. Let

$$f(x, y) = (y^2 + 2 + \sin y)x.$$

(a) Find the equation of the plane that is tangent to the graph of f at the point $(3, 0)$.
The formula for the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - a).$$

In this case, $a = 3$ and $b = 0$. We find $f(3, 0) = (0 + 2 + \sin 0)(3) = 6$. Now find the partial derivatives:

$$f_x(x, y) = y^2 + 2 + \sin y \qquad f_y(x, y) = 2yx + x \cos y.$$

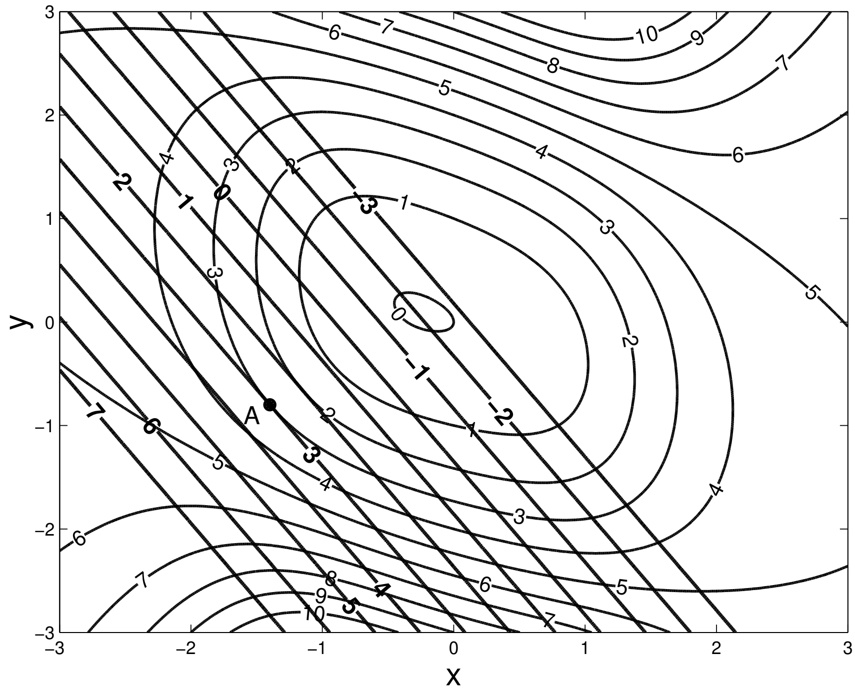
so $f_x(3, 0) = 2$ and $f_y(3, 0) = 3 \cos 0 = 3$. Then the equation of the tangent plane is

$$z = 6 + (2)(x - 3) + (3)(y - 0) = 2x + 3y.$$

(b) Use the local linearization of f near $(3, 0)$ to estimate $f(3.2, 0.1)$.
The tangent plane found in part (a) *is* the local linearization near $(3, 0)$, so we can use the result of part (a) to find

$$f(3.2, 0.1) \approx 2(3.2) + 3(0.1) = 6.7$$

6. The following is the contour diagram for a function $f(x, y)$.



Consider the tangent plane to the graph of f at the point labeled A.

(a) In the plot above, sketch the contour diagram of this tangent plane.

The straight lines show the contour lines for the tangent plane. The lines have been drawn for contour values -3, -2, . . . , 5, 6, 7.

(b) Describe and explain your answer to part (a). That is, describe the contours lines, and explain how you know how the curves are shaped and oriented.

The contour line of the tangent plane through A will be tangent to the contour line of f through A, and these two curves will have the same value 3. As for all planes, the contour lines of the tangent plane will be parallel lines that are evenly spaced. Since the tangent plane provides a good approximation to f when (x, y) is close to A, the spacing of the tangent plane's contour lines will be almost the same as the spacing of the contour lines of f near A.