Math 213 – Calculus III **Final Exam** **Spring 2002**

Name: ____

You must show your work and justify your answers to receive full credit.

#	Points	Score
1	16	
2	16	
3	17	
4	17	
5	17	
6	17	
Total	100	

Spherical Coordinates

$$x = \rho \sin\phi \cos\theta,$$
$$y = \rho \sin\phi \sin\theta,$$
$$z = \rho \cos\phi$$

1. Let
$$g(x, y, z) = \sqrt{2x^2 + y^2 + 4z^2}.$$

   (a) Describe the shapes of the level surfaces of g.

   (b) In three different graphs, sketch the three cross sections to the level surface $g(x, y, z) = 1$ for which

      i. $x = 0$,

      ii. $y = 0$,

      iii. $z = 0$.

   In each cross section, label the axes and any intercepts.

   (c) Find the equation of the plane tangent to the surface $g(x, y, z) = 1$ at the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right)$.

2. An assortment of TRUE/FALSE or short answer questions:

   (a) TRUE or FALSE: If $f(x, y)$ is defined for all (x, y), $\lim_{x \to 0} f(x, 0) = 0$, $\lim_{y \to 0} f(0, y) = 0$, and $f(0, 0) = 0$, then f is continuous at $(0, 0)$.

   (b) TRUE or FALSE: If f is differentiable at $(0, 0)$, then f is continuous at $(0, 0)$.

   (c) TRUE or FALSE: If f is a continuous function defined on the region $x^2 + y^2 \leq 9$, then f has a maximum value and a minimum value in this region.

   (d) TRUE or FALSE: If $f_x(0, 0)$ exists, and $f_y(0, 0)$ exists, then f is differentiable at $(0, 0)$.

   (e) TRUE or FALSE: If f is differentiable at $(0, 0)$, then the tangent plane to the graph of f at $(0, 0)$ is given by $z = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$.

   (f) Give an example of a function $f(x, y)$ for which $(0, 0)$ is a local minimum, but for which the second derivative test fails to determine this classification.

   (g) Give an example of a function $g(x, y)$ which is differentiable everywhere except along the line $y = x$.

   (h) Let $H(x, y) = x^2 - y^2 + xy$, and suppose that x and y are both functions that depend on t. Express $\dfrac{dH}{dt}$ in terms of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

3. Suppose f is a differentiable function such that

$$f(1,3) = 1, \quad f_x(1,3) = 2, \quad f_y(1,3) = 4,$$

$$f_{xx}(1,3) = 2, \quad f_{xy}(1,3) = -1, \quad \text{and} \quad f_{yy}(1,3) = 4.$$

(a) Find $\text{grad} f(1,3)$.

(b) Find a vector in the plane that is perpendicular to the contour line $f(x,y) = 1$ at the point $(1,3)$.

(c) Find a vector that is perpendicular to the surface $z = f(x,y)$ (i.e. the graph of f) at the point $(1,3,1)$.

(d) At the point $(1,3)$, what is the rate of change of f in the direction $\vec{i} + \vec{j}$?

(e) Use a quadratic approximation to estimate $f(1.2, 3.3)$.

4. Let

$$f(x,y) = x^2 - 4x + y^2 - 4y + 16.$$

(a) Find and classify the critical points of f.

(b) Find the maximum and minimum values of f subject to the constraint

$$x^2 + y^2 = 18$$

(c) Find the maximum and minimum values of f subject to the constraint

$$x^2 + y^2 \leq 18$$

(d) Approximate the maximum value of f subject to the constraint

$$x^2 + y^2 = 18.3$$

(Explain your answer in terms of Lagrange multipliers.)

5. Suppose the integral of some function f over a region R in the plane is given in polar coordinates as

$$\int_0^3 \int_0^{\frac{\pi}{2}} r^2 \, d\theta dr.$$

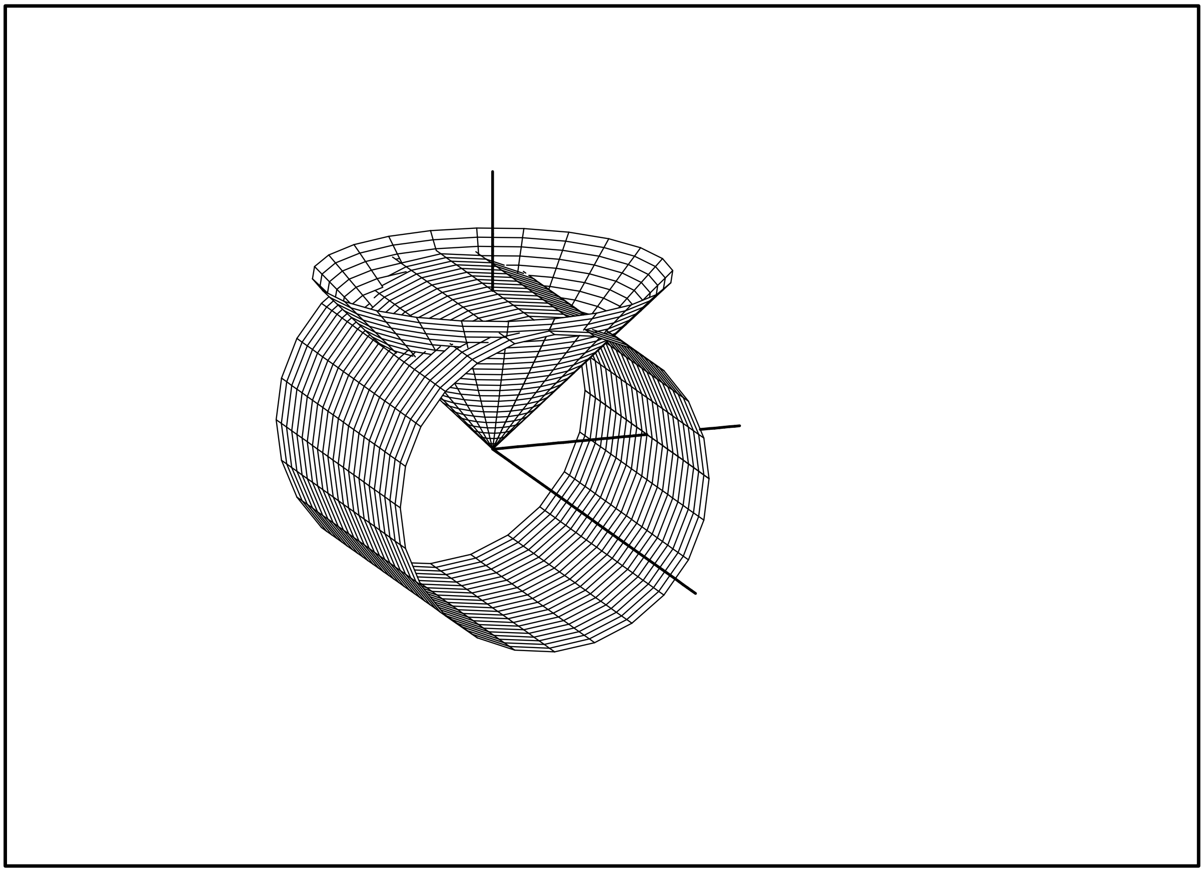
(a) Sketch the region of integration R in the xy plane.

(b) Convert this integral to Cartesian coordinates.

(c) Evaluate the integral. (You may use either polar or Cartesian coordinates.)

(d) What is the average value of f on the region R?

6. Let W be the solid region above the cone $z = \sqrt{x^2 + y^2}$ and inside the cylinder $y^2 + z^2 = 1$.



Express the volume of W as a triple integral in

(a) spherical coordinates, and

(b) Cartesian coordinates.

Do not evaluate the integrals!