Solving The "Coffee Cup Problem"

The differential equation for $H(t)$, the temperature at time t, is

$$\frac{dH}{dt} = k(70 - H).$$

Note that

$$H(t) = 70 \tag{1}$$

(i.e. $H(t)$ is the *function with constant value 70*) is a solution to the differential equation. Now assume that $H(t) \neq 70$. We treat $\frac{dH}{dt}$ as if it really is a fraction, and rewrite the differential equation as

$$\frac{dH}{H - 70} = -k\,dt,$$

and then integrate both sides:

$$\int \frac{dH}{H - 70} = -\int k\,dt.$$

This gives us

$$\ln|H - 70| = -kt + C_1.$$

Note the absolute value on the left; recall from Calculus that $\int \frac{dx}{x} = \ln|x| + c$. Now exponentiate both sides to obtain

$$|H - 70| = e^{-kt+C_1} = e^{C_1}e^{-kt} = C_2e^{-kt},$$

where $C_2 = e^{C_1}$. (Note that $C_2 > 0$.) Now, if $H - 70 > 0$, then $|H - 70| = H - 70$, and we have

$$H(t) = 70 + C_2e^{-kt}. \tag{2}$$

If $H - 70 < 0$, then $|H - 70| = -(H - 70)$, and so

$$H(t) = 70 - C_2e^{-kt}. \tag{3}$$

Thus, the solutions to the differential equation are given by (1), (2), and (3). For this example, we see that we can combine these three formulas into one expression that includes all the solutions:

$$H(t) = 70 + C_3e^{-kt},$$

where C_3 is an arbitrary constant.

If we are given an initial condition

$$H(0) = H_0,$$

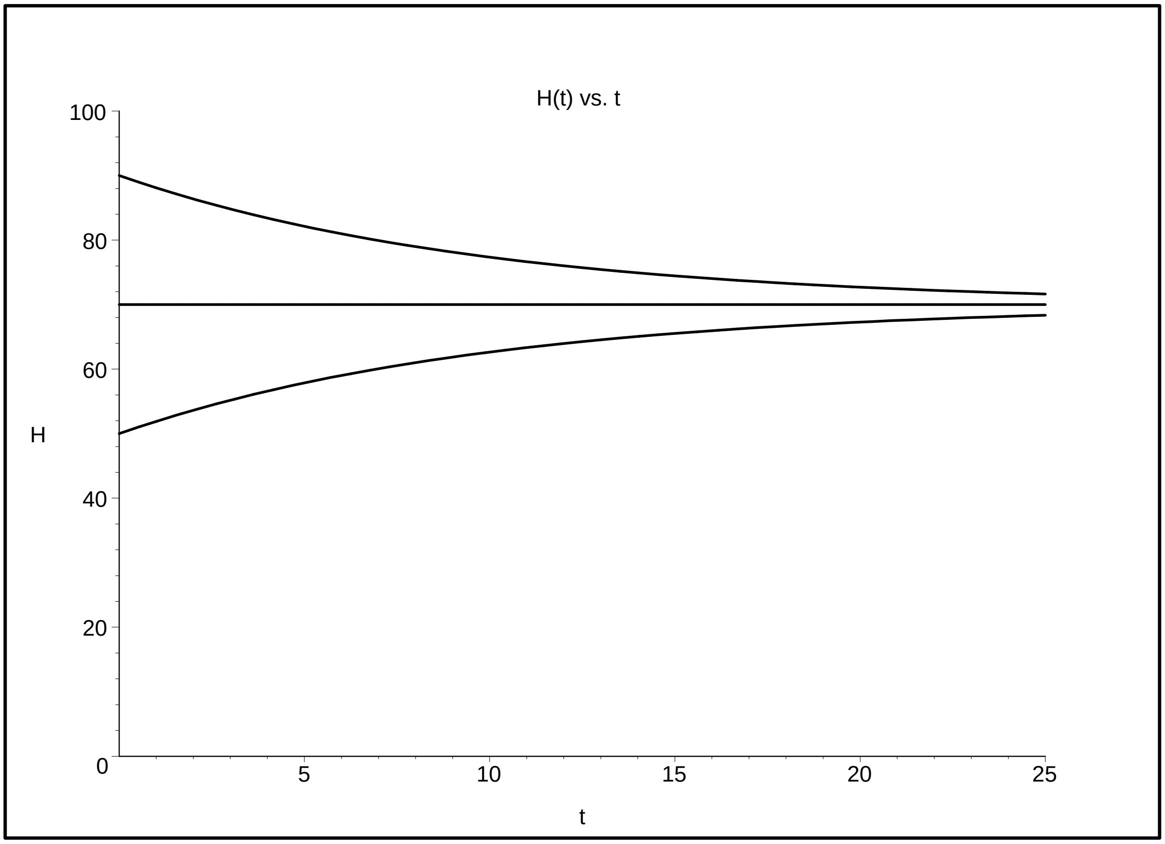
we can find C_3 in terms of H_0 by evaluating the solution at $t = 0$ and setting it equal to H_0:

$$H(0) = 70 + C_3e^0 = 70 + C_3 = H_0,$$

so $C_3 = H_0 - 70$, and the solution to the initial value problem is

$$H(t) = 70 + (H_0 - 70)e^{-kt}.$$

The following plot shows the solutions for $H_0 = 90$, $H_0 = 70$, and $H_0 = 50$, with $k = 0.1$, for the time interval $0 \leq t \leq 25$.


H(t) vs. t
100
80
60
H
40
20
0
5
10
15
20
25
t