Solutions to 4.3/2 and 4.3/10

4.3/2. The equation is

$$\frac{d^2y}{dt^2} + 9y = 5\sin 2t.$$

First find y_h. The homogeneous equation is $\frac{d^2y}{dt^2} + 9y = 0$, so when we guess $y_h = e^{st}$, we find $s^2 + 9 = 0$. Thus $s = \pm 3i$, and we have

$$y_h = k_1 \cos 3t + k_2 \sin 3t.$$

Now we find the particular solution y_p. The first guess is $y_p = A\cos 2t + B\sin 2t$, and neither term in y_p solves the homogeneous equation, so this will work. We find

$$\begin{aligned} y_p' &= -2A\sin 2t + 2B\cos 2t \\ y_p'' &= -4A\cos 2t - 4B\sin 2t. \end{aligned}$$

Now put these into the original equation:

$$\begin{gathered} -4A\cos 2t - 4B\sin 2t + 9A\cos 2t + 9B\sin 2t = 5\sin 2t \\ (5A)\cos 2t + (5B)\sin 2t = 5\sin 2t, \end{gathered}$$

and this implies $A = 0$ and $B = 1$. Thus $y_p = \sin 2t$. The general solution is

$$y(t) = y_h + y_p = k_1\cos 3t + k_2 \sin 3t + \sin 2t.$$

4.3/10. The initial value problem is

$$\frac{d^2y}{dt^2} + 4y = 3\cos 2t, \quad y(0) = 0, \quad y'(0) = 0.$$

First find y_h; following the usual procedure, we find $y_h = k_1\cos 2t + k_2 \sin 2t$.

Now we find the particular solution y_p. Our first guess is $y_p = A\cos 2t + B\sin 2t$. This will *not* work, because each term in this guess also solves the homogeneous problem. So we multiply by t to get the next guess: $y_p = At\cos 2t + Bt\sin 2t$. Neither term solves the homogeneous problem, so this will work. We find

$$\begin{aligned} y_p' &= (B - 2At)\sin 2t + (A + 2Bt)\cos 2t \\ y_p'' &= 4B\cos 2t - 4At\cos 2t - 4A\sin 2t - 4Bt\sin 2t. \end{aligned}$$

Put these into the original equation:

$$4B\cos 2t - 4At\cos 2t - 4A\sin 2t - 4Bt\sin 2t + 4At\cos 2t + 4Bt\sin 2t = 3\cos 2t,$$

so

$$4B\cos 2t - 4A\sin 2t = 3\cos 2t,$$

and we have $A = 0$ and $B = 3/4$. Thus $y_p = \frac{3}{4}t\sin 2t$.

The general solution is then

$$y(t) = y_h + y_p = k_1\cos 2t + k_2\sin 2t + \frac{3}{4}t\sin 2t.$$

Now choose k_1 and k_2 to satisfy the initial conditions. $y(0) = 0 \implies k_1 + 0 + 0 = 0 \implies k_1 = 0$. Now

$$y'(t) = 2k_2\cos 2t + (3/2)t\cos 2t + (3/4)\sin 2t,$$

and $y'(0) = 0 \implies 2k_2 + 0 + 0 = 0 \implies k_2 = 0$. The solution to the initial value problem is

$$y(t) = \frac{3}{4}t\sin 2t.$$