Solutions to 5.3/11 and 5.3/13

5.3/11. The system is

$$\frac{dx}{dt} = x - 3y^2,$$
$$\frac{dy}{dt} = -y.$$

A Hamiltonian system has the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y},$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

for some function $H(x, y)$. So we want to know if there is a function $H(x, y)$ such that $\frac{\partial H}{\partial x} = y$ and $\frac{\partial H}{\partial y} = x - 3y^2$. If there is such a function, it must satisfy $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$. This is the test we use for a Hamiltonian system. In this case, we have

$$\frac{\partial^2 H}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial y}(y) = 1,$$

and

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial}{\partial x}(x - 3y^2) = 1.$$

Since these are equal, the system is Hamiltonian.

To find the Hamiltonian function, we can start with $\frac{\partial H}{\partial x} = y$ and integrate with respect to x. This gives us $H(x, y) = xy + \phi(y)$. To find $\phi(y)$, we now compute $\frac{\partial H}{\partial y}$ and set it equal to $x - 3y^2$:

$$\frac{\partial H}{\partial y} = x + \phi'(y) = x - 3y^2.$$

This implies that $\phi'(y) = -3y^2$, so $\phi(y) = -y^3 + C$. The constant C does not change the shapes of the level curves of H, so we might as well choose it to be zero. Thus, the Hamiltonian function is

$$H(x, y) = xy - y^3.$$

5.3/13. The system is

$$\frac{dx}{dt} = x \cos y,$$
$$\frac{dy}{dt} = -y \cos x.$$

In this case, $\frac{\partial}{\partial x}(x \cos y) = \cos y$, and $-\frac{\partial}{\partial y}(-y \cos x) = \cos x$. The mixed partial derivatives are not equal, so this system is not Hamiltonian.