Solved Problems

Section 1.2/ 8, 10, 12, 28, 33, 35

1.2/8. I'm going to do this one in gory detail–you don't have to do all these steps explicitly every time you solve a problem like this. First, we note that $y(t) \equiv 2$ is an equilibrium solution. Now suppose $y(t) \neq 2$; in fact, let's assume that $y(t) < 2$ Then

$$\frac{dy}{dt} = 2 - y \implies \frac{dy}{2-y} = dt \implies -\ln|2-y| = t + C_1 \implies |2-y| = e^{-t-C_1} = C_2 e^{-t}.$$

(Note that $C_2 = e^{-C_1} > 0$.) Now, since $y(t) < 2$, we have $2 - y > 0$, thus $|2-y| = 2-y$. Thus

$$|2-y| = C_2 e^{-t} \implies 2 - y = C_2 e^{-t} \implies y = 2 - C_2 e^{-t}.$$

Suppose, on the other hand, that $y(t) > 2$; then $2 - y < 0$, so $|2-y| = -(2-y)$. Then

$$|2-y| = C_2 e^{-t} \implies -(2-y) = C_2 e^{-t} \implies y = 2 + C_2 e^{-t}$$

The two cases ($y(t) < 2$ and $y(t) > 2$) have given us corresponding families of solutions,

$$y = 2 - C_2 e^{-t}$$

and

$$y = 2 + C_2 e^{-t},$$

where in each case, the constant C_2 is positive. We can combine the two cases into

$$y = 2 + C_3 e^{-t}$$

where C_3 is an arbitrary constant (possibly positive or negative). In fact, by setting $C_3 = 0$, we obtain the equilibrium solution, so this formula gives us *all* the solutions.

1.2/10. First note that $y(t) \equiv 0$ is an equilibrium solution. Now suppose $y \neq 0$:

$$\frac{dy}{dt} = (ty)^2 = t^2 y^2 \implies \frac{dy}{y^2} = t^2 dt \implies \frac{-1}{y} = t^3/3 + C_1 \implies y = -\frac{1}{t^3/3 + C_1} = -\frac{3}{t^3 + C_2}.$$

So the solutions are

$$y(t) = -\frac{3}{t^3 + C_2},$$

where C_2 is an arbitrary constant, *or*

$$y(t) = 0.$$

1.2/12. $y(t) \equiv 0$ is an equilibrium solution. If $y(t) \neq 0$, $\frac{dy}{dt} = t\sqrt[3]{y} = ty^{1/3} \implies \frac{dy}{y^{1/3}} = tdt \implies \frac{3y^{2/3}}{2} = t^2/2 + C_1 \implies y^{2/3} = t^2/3 + C_2 \implies y^2 = (t^2/3 + C_2)^3 \implies y = \pm\sqrt{(t^2/3 + C_2)^3}$. Thus the solutions are

$$y(t) = 0, \quad \text{or} \quad y(t) = \sqrt{(t^2/3 + C_2)^3}, \quad \text{or} \quad y(t) = -\sqrt{(t^2/3 + C_2)^3}.$$

1.2/26. $y(t) \equiv 0$ is an equlibrium solution. If $y(t) \neq 0$,

$$\frac{dy}{dt} = ty^2 + 2y^2 = (t+2)y^2 \implies \frac{dy}{y^2} = (t+2)dt \implies -\frac{1}{y} = t^2/2 + 2t + C_1 \implies y = -\frac{1}{t^2/2 + 2t + C_1}.$$

We require $y(0) = 1$, which gives us $y(0) = -\frac{1}{C_1} = 1 \implies C_1 = -1$. Thus

$$y(t) = -\frac{1}{t^2/2 + 2t - 1} = \frac{-2}{t^2 + 4t - 2}.$$

1.2/28. We see that $x(t) \equiv 0$ is an equilibrium solution. Now if $x(t) \neq 0$,

$$\frac{dx}{dt} = -xt \implies \frac{dx}{x} = -tdt \implies \ln|x| = -t^2/2 + C_1 \implies |x| = e^{-t^2/2 + C_1} = C_2 e^{-t^2/2} \implies x = C_3 e^{-t^2/2}.$$

Thus the general solution is

$$x(t) = C_3 e^{-t^2/2},$$

where C_3 is an arbitrary constant. We want the solution for which $x(0) = 1/\sqrt{\pi}$, so we have

$$x(0) = C_3 e^0 = C_3 = 1/\sqrt{\pi}.$$

So the solution to the initial value problem is

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}.$$

1.2/33. First, recall from Calculus that $\displaystyle\int \frac{dy}{y^2+1} = \arctan(y)$. Then

$$\frac{dy}{dt} = (y^2+1)t \implies \frac{dy}{y^2+1} = tdt \implies \arctan(y) = t^2/2 + C_1 \implies y = \tan\left(t^2/2 + C_1\right).$$

We want the solution where $y(0) = 1$, so we must have $y(0) = \tan(C_1) = 1 \implies C_1 = \pi/4$. Thus the solution to the initial value problem is

$$y(t) = \tan\left(t^2/2 + \pi/4\right).$$

1.2/35. Let $S(t)$ be the amount of salt (measured in pounds) in the bucket at time t, where t is the time (in minutes). There are two processes here that are causing $S(t)$ to change. First, we are dumping salt into the bucket at the rate 1/4 lbs/min; this will cause the amount of salt to increase at the rate 1/4 lbs/min.

The amount of salt is also changing because of the water that is leaving through the spigot. We are told that the water leaves at the rate 1/2 gal/min. This water contains salt, so this causes the amount of salt to decrease, but at what rate? The *concentration* of the salt in the water is $S(t)/5$ lbs/gal (we are told that the bucket holds five gallons). The rate at which the salt is leaving is given by the formula (*concentration*)$\times$(*flow rate*), which in this case gives us $(S(t)/5) \times (1/2) = S(t)/10$.

Thus, the bucket is gaining salt at the rate of 1/4 lbs/min, but is also losing it at the rate of $S(t)/10$ lbs/min. Therefore, the *net rate of change* of the salt is $1/4 - S/10$, and the corresponding mathematical version of that statement is

$$\frac{dS}{dt} = \frac{1}{4} - \frac{S}{10}.$$

We are told that initially the bucket is full of pure water, which means

$$S(0) = 0.$$

We now have an initial value problem to solve. The differential equation is separable. Here we go: $\dfrac{dS}{dt} = \dfrac{1}{4} - \dfrac{S}{10} \implies \dfrac{dS}{\frac{1}{4} - \frac{S}{10}} = dt \implies \dfrac{10dS}{2.5 - S} = dt \implies -10\ln|2.5 - S| = t + C_1 \implies |2.5 - S| = e^{-(t+C_1)/10} = C_2e^{-t/10} \implies 2.5 - S = C_3e^{-t/10} \implies S = 2.5 - C_3e^{-t/10}$. So the general solution to the differential equation is

$$S(t) = 2.5 - C_3e^{-t/10}.$$

We want the solution where $S(0) = 0$, so

$$S(0) = 2.5 - C_3 = 0 \implies C_3 = 2.5.$$

The solution to the initial value problem is

$$S(t) = 2.5 - 2.5e^{-t/10}.$$

To answer parts (a)-(d), we just plug in numbers: (a) $S(1) \approx 0.2379$; (b) $S(10) = 1.580$; (c) $S(60) = 2.494$; (d) $S(1000) = 2.500$.

To answer (e), we can take "a very, very long time" to mean $t \to \infty$. Then

$$\text{(e)} \qquad \lim_{t\to\infty} S(t) = \lim_{t\to\infty} 2.5 - 2.5e^{-t/10} = 2.5.$$