Undetermined Coefficients: More Examples

Example. Find the correct form of the particular solution to

$$y'' + 4y = t\sin(2t)$$

but do not solve for the coefficients.

Solution. First we find $y_h(t) = k_1\cos(2t) + k_2\sin(2t)$. Since $g(t) = t\sin(2t)$, our first guess for y_p is $(A_1t + A_0)\cos(2t) + (B_1t + B_0)\sin(2t)$. Since the terms $A_0\cos(2t)$ and $B_0\sin(2t)$ solve the homogeneous equation, we must multiply our initial guess by t to obtain

$$\begin{aligned} y_p(t) &= t\{(A_1t + A_0)\cos(2t) + (B_1t + B_0)\sin(2t)\} \\ &= A_1t^2\cos(2t) + A_0t\cos(2t) + B_1t^2\sin(2t) + B_0t\sin(2t). \end{aligned}$$

None of the terms in y_p solve the homogeneous equation, so this will work.

Example. Find the general solution to

$$y'' + 4y' + 4y = te^{-2t}.$$

Solution. The characteristic polynomial is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$, so the only root is $\lambda_1 = -2$. The solution to the homogeneous equation is then $y_h(t) = k_1e^{-2t} + k_2te^{-2t}$. Now $g(t) = te^{-2t}$, so our first guess is $(A_1t + A_0)e^{-2t}$, but both A_1te^{-2t} and A_0e^{-2t} solve the homogeneous equation. Therefore we multiply by t to obtain $t(A_1t + A_0)e^{-2t} = A_1t^2e^{-2t} + A_0te^{-2t}$. However, the second term, A_0te^{-2t}, is still a solution to the homogeneous equation, so multiply by t again. We obtain

$$\begin{aligned} y_p(t) &= t^2\left\{A_1t + A_0)e^{-2t}\right\} \\ &= A_1t^3e^{-2t} + A_0t^2e^{-2t}, \end{aligned}$$

and this guess will work.

To find the coefficients, we will need

$$\begin{aligned} y_p'(t) &= -2(A_1t^3 + A_0t^2)e^{-2t} + (3A_1t^2 + 2A_0t)e^{-2t} \\ &= (-2A_1t^3 + (3A_1 - 2A_0)t^2 + 2A_0t)e^{-2t} \end{aligned}$$

and

$$\begin{aligned} y_p''(t) &= -2(-2A_1t^3 + (3A_1 - 2A_0)t^2 + 2A_0t)e^{-2t} + (-6A_1t^2 + (6A_1 - 4A_0)t + 2A_0)e^{-2t} \\ &= (4A_1t^3 + (-12A_1 + 4A_0)t^2 + (6A_1 - 8A_0)t + 2A_0)e^{-2t} \end{aligned}$$

To solve for the coefficients, put these into the differential equation. The exponential factor in each term can be canceled, and we are left with

$$(4A_1 - 8A_1 + 4A_1)t^3 + (-12A_1 + 4A_0 + 12A_1 - 8A_0 + 4A_0)t^2 + (6A_1 - 8A_0 + 8A_0)t + 2A_0 = t.$$

The coefficients of t^3 and t^2 are automatically zero. To make the remaining terms match for all t, we must have $6A_1 = 1$ and $2A_0 = 0$. Thus $A_0 = 0$ and $A_1 = 1/6$, and $y_p(t) = (1/6)t^3e^{-2t}$.

The general solution is

$$y(t) = y_h(t) + y_p(t) = k_1e^{-2t} + k_2te^{-2t} + \frac{1}{6}t^3e^{-2t}.$$