Homework 1 Solutions

Text Problems

1.1/4.

(a) The statement of the problem says t is time, but it does not specify the starting point. I will define t to be the number of years since 1939. Then the initial condition, given by the first entry in the table, is $A(0) = 32800$.

As shown in Section 1.1 (and subsequently shown in class), the solution to the differential equation is

$$A(t) = A_0 e^{kt},$$

where A_0 is a constant. To find A_0, we use the initial value:

$$A(0) = A_0 e^0 = A_0 = 32800.$$

so the solution to the initial value problem is

$$A(t) = 32800 e^{kt}.$$

(b) If the data agrees with the model, we can use any data point (except for the first) to determine k. I will try the model with two values of k, determined by two different data points.

To find k_1, I will use the second data point: $A(5) = 55800$. Thus

$$A(5) = 32800 e^{5k_1} = 55800 \implies e^{5k_1} = \frac{55800}{32800} \implies k_1 = \frac{1}{5} \ln\left(\frac{55800}{32800}\right) = 0.10626907\ldots$$

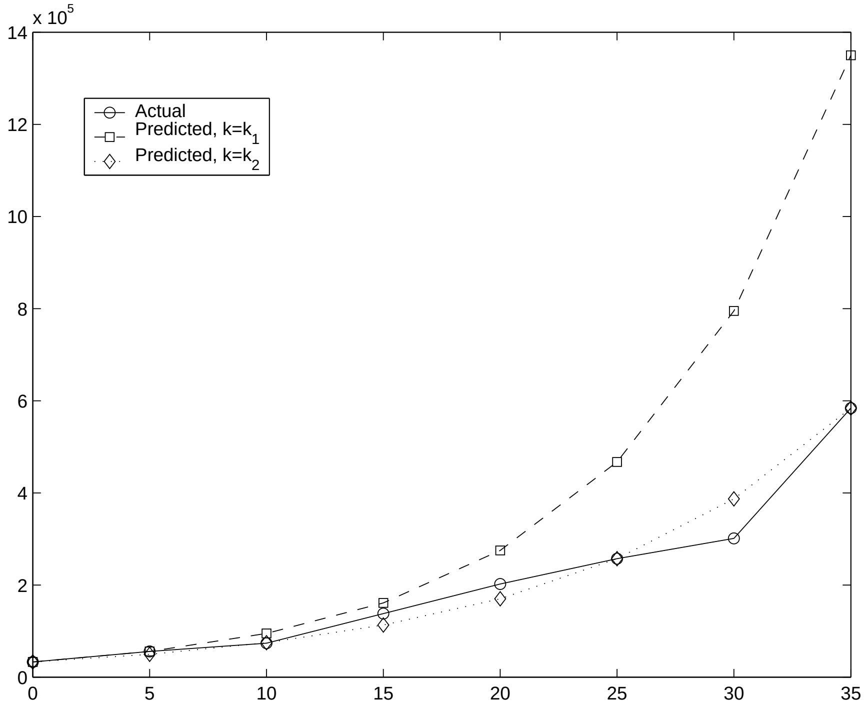
To find k_2, I use the last data point, $A(35) = 584000$. We find

$$k_2 = \frac{1}{35} \ln\left(\frac{58400}{32800}\right) = 0.082270641\ldots$$

(c) Here is a table that shows that actual data, and the predictions of the model with $k = k_1$ and $k = k_2$. Only three significant digits are shown for the predictions.

Year	t	Actual	Predicted with $k = k_1$	Predicted with $k = k_2$
1939	0	32,800	32,800	32,800
1944	5	55,800	55,800	49,500
1949	10	73,600	94,900	74,700
1954	15	138,000	161,000	113,000
1959	20	202,000	275,000	170,000
1964	25	257,000	467,000	257,000
1969	30	301,000	795,000	387,000
1974	35	584,000	1,350,000	584,000
2010	71		6.20×10^7	1.13×10^7
2050	111		4.35×10^9	3.03×10^8
2100	161		8.84×10^{11}	1.86×10^{10}

(d) The following graph compares the actual data with the models for $0 \le t \le 35$.



With $k = k_1$, the predictions are significantly larger than the the actual data. The model does not appear very good, and it would not be prudent to believe the predictions for 2010, 2050, and 2100, especially since the model gives areas that exceed the area of Australia!

With $k = k_2$, the model appears more reasonable for the time $0 \le t \le 35$. However, this model also predicts that the area will exceed the area of Australia by 2010, so we still can't believe the predictions.

It appears that the equation $\frac{dA}{dt} = kA$ is not a good model. A model based on the logistic equation might be better.

1.1/8.

(a) We will use the notation given and translate the verbal description of the behavior into a mathematical equation. First, "the rate at which a quantity of a radioactive isotope decays" refers to $\frac{dr}{dt}$; the word "decay" suggests that we expect the rate of change of $r(t)$ to be negative. The description says that the rate of change is proportional to the amount present, which is just $r(t)$. The proportionality constant is what the problem statement calls "$-\lambda$". Thus the equation is

$$\frac{dr}{dt} = -\lambda r(t).$$

(b) The statement "the amount of the isotope present at $t = 0$ is r_0" translates directly into $r(0) = r_0$, so the initial value problem is

$$\frac{dr}{dt} = -\lambda r(t), \qquad r(0) = r_0.$$

1.1/9. First, some general comments about half-life. We know that the amount of an isotope is determined by the equation $dr/dt = -\lambda r(t)$. The solution to this first order differential equation is $r(t) = r_0 e^{-\lambda t}$, where (as in the previous problem) r_0 is the amount present at $t = 0$. Let T be the half-life of the isotope;

this means that it takes T units of time for a quantity of the isotope to decay to half its original amount. Mathematically, this says $r(T) = r(0)/2 = r_0/2$. But $r(T) = r_0 e^{-\lambda T}$, so T must satisfy

$$r_0 e^{-\lambda T} = \frac{r_0}{2},$$

or, canceling r_0,

$$e^{-\lambda T} = \frac{1}{2}.$$

In parts (a) and (b) of this question, we are given T and asked to find λ. By solving the above equation for λ, we obtain

$$\lambda = -\frac{\ln\left(\frac{1}{2}\right)}{T} = \frac{\ln 2}{T}.$$

(a) $\lambda = \dfrac{\ln 2}{5230} \approx 1.325 \times 10^{-4}$.

(b) $\lambda = \dfrac{\ln 2}{8} \approx 0.08664$.

(c) In (a), the units of λ are year^{-1} (in other words, "per year"), and in (b), the units of λ are day^{-1}.

(d) We will get the same answer. Note that in the comments above, r_0 drops out of the equation, so the half-life does not depend on the initial amount.

1.1/12.

(a) From Exercise 9, we know that the half-life of I-131 is 8 days, and the decay-rate parameter is $\lambda = 0.08664$ day^{-1}. Since $r(t) = r_0 e^{-\lambda t}$, where $r_0 = r(0)$, the fraction remaining after t days is $r(t)/r(0) = e^{-\lambda t}$. In three days (i.e. 72 hours), the fraction remaining is

$$\frac{r(3)}{r(0)} = e^{-(0.08664)(3)} \approx 0.7716,$$

so there will be about 77 percent remaining.

(b) The fraction remaining after 5 days (72 hours for shipping, plus 48 hours of storage) is

$$\frac{r(5)}{r(0)} = e^{-(0.08664)(5)} \approx 0.6491,$$

so there will be about 65 percent remaining.

(c) Since $e^{\lambda t} \neq 0$ for any t, according to the mathematical model, the I-131 will never completely decay.

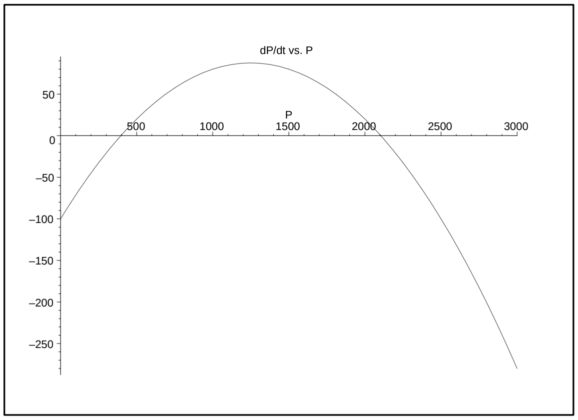
In practice, the amount remaining will eventually be so small that any remaining radioactivity will be less than the background radiation to which we are continuously exposed. However, I am not a radiologist, so I can't say when the remaining I-131 "can be thrown away without special precautions."

1.1/14.

(a) In this case, the model is

$$\begin{aligned}\frac{dP}{dt} &= k\left(1 - \frac{P}{N}\right)P - 100\\ &= 0.3\left(1 - \frac{P}{2500}\right)P - 100\end{aligned}$$

Let's plot dP/dt vs. P:

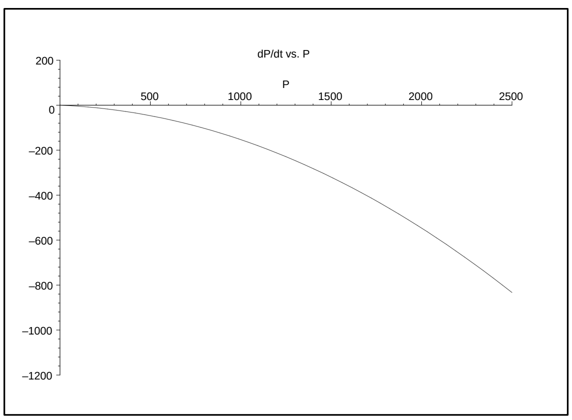


The equilibria are $P \approx 396$ and $P \approx 2104$. For $P > 2104$, we see that $dP/dt < 0$, which tells us that P will *decrease*. In fact, $P(t)$ will continue to decrease towards the equilibrium at $P \approx 2104$. In other words, in the long-term we expect the population to level off at about 2104.

(b) In this case, the model is

$$\begin{aligned}\frac{dP}{dt} &= k\left(1 - \frac{P}{N}\right)P - \frac{P}{3} \\ &= 0.3\left(1 - \frac{P}{2500}\right)P - \frac{P}{3}\end{aligned}$$

and the plot of dP/dt vs. P is:



The equilibria are $P = 0$ and $P = -2500/9 \approx -278$. Of course, negative values of P are not relevant for the model, but this does tell us that $dP/dt < 0$ for all $P > 0$. This means that for *any* starting population $P(0)$ (including the case $P(0) = 2500$), $P(t)$ will decrease towards zero. In the long-term, the population will die out.

1.2/ 16.
Separate:

$$y\,dy = t\,dt$$

Integrate:

$$\frac{y^2}{2} = \frac{t^2}{2} + C$$

Solve for $y(t)$:

$$y = \pm\sqrt{t^2 + 2C}.$$

Note that this means there are two families of solutions: $y = \sqrt{t^2 + 2C}$ and $y = -\sqrt{t^2 + 2C}$. If an initial condition were given, it would determine the sign of the solution, and the value of C.

1.2/ 18.
We have

$$\frac{dy}{dt} = \frac{1}{ty + t + y + 1} = \frac{1}{(t+1)(y+1)} = \left(\frac{1}{t+1}\right)\left(\frac{1}{y+1}\right).$$

Now separate:

$$(y+1)\,dy = \frac{dt}{t+1}$$

Integrate:

$$\frac{y^2}{2} + y = \ln|t+1| + C_1$$

Solve for y:

$$\frac{y^2}{2} + y - \ln|t+1| - C_1 = 0$$
$$y = -1 \pm \sqrt{1 + 2\ln|t+1| + 2C_1}$$

or

$$y = -1 \pm \sqrt{2\ln|t+1| + C_2}$$

where $C_2 = 1 + 2C_1$.

1.2/ 26.
We have

$$\frac{dy}{dt} = ty^2 + 2y^2 = (t+2)y^2$$

Separate:

$$\frac{dy}{y^2} = (t+2)\,dt$$

Integrate:

$$-\frac{1}{y} = \frac{t^2}{2} + 2t + C$$

We can use the initial condition $y(0) = 1$ (that is, $y = 1$ when $t = 0$) to determine C:

$$-\frac{1}{1} = 0 + 0 + C \implies C = -1$$

So we have

$$-\frac{1}{y} = \frac{t^2}{2} + 2t - 1$$

Solve for y:

$$y = \frac{-1}{\frac{t^2}{2} + 2t - 1} = \frac{-2}{t^2 + 4t - 2}$$

1.2/ 34.
Separate:

$$(2y + 3)\, dy = dt$$

Integrate:

$$y^2 + 3y = t + C$$

Use the initial condition $y(0) = 1$ to find C:

$$1 + 3 = 0 + C \implies C = 4$$

So we have

$$y^2 + 3y = t + 4$$

Solve for y:

$$\begin{aligned} y^2 + 3y - t - 4 &= 0 \\ y &= \frac{-3 \pm \sqrt{9 + 4(t + 4)}}{2} \\ &= \frac{-3 \pm \sqrt{4t + 25}}{2} \end{aligned}$$

This is still *two* possible solutions. We again use the initial condition to determine which is the correct solution. Since $y(0) = 1$, we must choose the *positive* square root. The solution to the initial value problem is

$$y(t) = \frac{-3 + \sqrt{4t + 25}}{2}$$

Additional Problems

1.

(a) Since the air is being pumped *into* the balloon at 10 cubic centimeters per second, the volume of the balloon must be increasing at this rate. So, if the volume v is measured in cubic centimeters, and if t is time, measured in seconds, we have

$$\frac{dv}{dt} = 10.$$

This is the differential equation for v.

(b) Take the t derivative of both sides of the formula $v = 4\pi r^3/3$. This gives

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

But, as we showed in (a), $dv/dt = 10$, so

$$4\pi r^2 \frac{dr}{dt} = 10$$

Solving for dr/dt gives the differential equation for r:

$$\frac{dr}{dt} = \frac{10}{4\pi r^2}.$$

(c) Solving the equation for v is certainly easy enough:

$$v(t) = 10t + C.$$

The differential equation for r is separable, and we could following the usual procedure to solve it. Or, we can solve $v = 4\pi r^3/3$ for r:

$$r = \left(\frac{3v}{4\pi}\right)^{\frac{1}{3}}$$

Now put in $v(t)$:

$$r(t) = \left(\frac{3(10t + C)}{4\pi}\right)^{\frac{1}{3}}$$

This is the solution to the differential equation for r.

2.

(a) The equation for $a(t)$ is $\frac{da}{dt} = -ka$, and by now we know that the solution is $a(t) = a_0 e^{-kt}$, where $a(0) = a_0$. We are told that the patient is given 5 units at $t = 0$, so $a(0) = 5$. Therefore, $a(t) = 5e^{-kt}$. We are also told that $a(24) = 1.2$. We use this to find k:

$$a(24) = 5e^{-24k} = 1.2 \implies -24k = \ln\left(\frac{1.2}{5}\right)$$

so

$$k = -\frac{1}{24}\ln\left(\frac{1.2}{5}\right) = 0.05946318\ldots$$

(b) Immediately after the second dose, the patient has 6.2 units of drug in the bloodstream: 1.2 units are left over from the initial dose, and 5 more have been added by the second dose. We can now redefine time so that $t = 0$ is the time when the second dose was administered. Then we have $a(0) = 6.2$, so $a(t) = 6.2e^{-kt}$, where k is the same as before. Then 24 hours later, we have $a(24) = 6.2e^{(-0.05946)(24)} \approx 1.488$. Thus, at the end of the second 24 hours, there are 1.488 units of the drug in the bloodstream.

(c) Let a_i be the amount of drug in the bloodstream at the end of day i, just before the next dose of 5 units is given. We were told $a_1 = 1.2$, and in (b) we computed $a_2 = 1.488$. To compute a_3, a_4, etc., we follow the same procedure that we used in (b):

$$a_3 = (5 + 1.488)e^{-(0.05946)(24)} = 1.557,$$
$$a_4 = (5 + 1.557)e^{-(0.05946)(24)} = 1.574,$$

and we see that the pattern is

$$a_i = (5 + a_{i-1})e^{-(0.05946)(24)}.$$

Continuing with the calculations, we find

$$a_5 = (5 + 1.574)e^{-(0.05946)(24)} = 1.578,$$
$$a_6 = (5 + 1.578)e^{-(0.05946)(24)} = 1.579,$$
$$a_7 = (5 + 1.579)e^{-(0.05946)(24)} = 1.579.$$

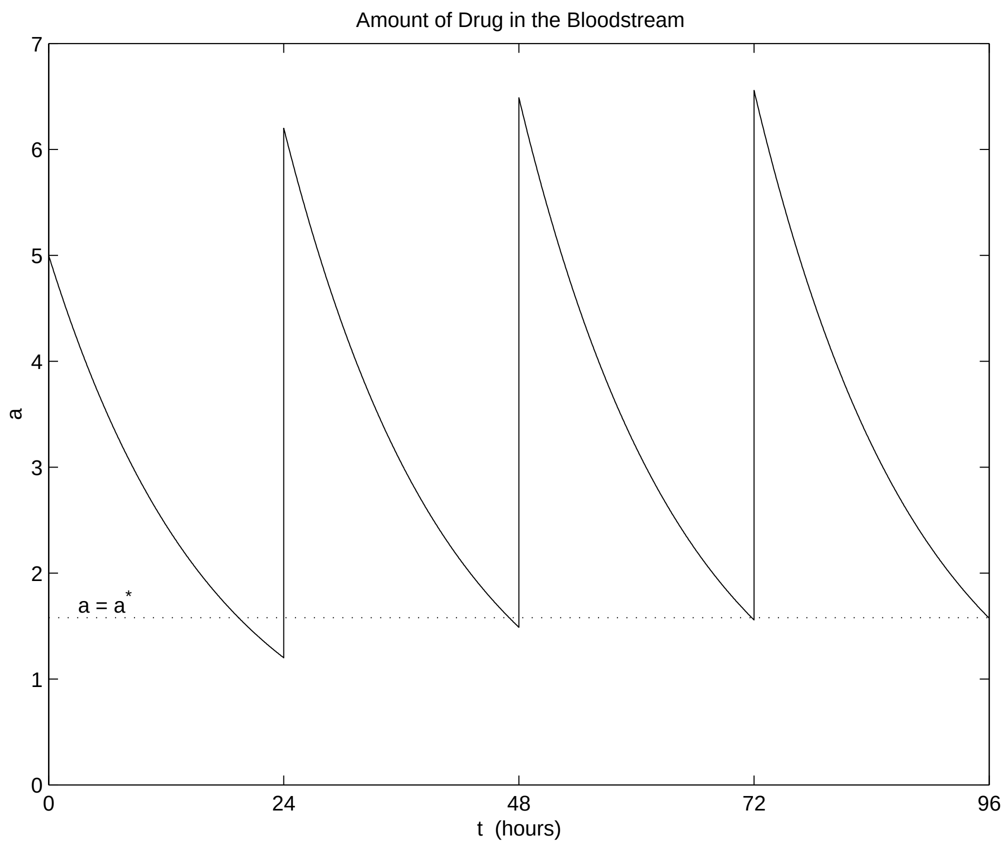
Thus, when we keep only four significant digits in our calculations, the amount of the drug in the bloodstream just before each dose converges to the value 1.579. If we kept more digits, the value would continue to increase, but it would approach a limiting value. We can check this by using the equation for a_i given above. We look for a "fixed point" of this equation; that is, a value where $a_i = a_{i-1}$. If we call this value a^*, the equation becomes

$$a^* = (5 + a^*)e^{-(0.05946)(24)}$$

and solving for a^* gives $a^* = 1.579106\ldots$.

In terms of the original problem, $(5 + a^*)$ is precisely the amount of the drug for which the amount "cleared" during a 24 hour period is 5 units.

The following graph shows the amount of the drug in the bloodstream during the first four days. The dotted line shows $a^* = 1.579106$.



Note: It is possible to find a formula for a_i explicity in terms of i, rather than as the recursion relation given on the previous page. If $r = e^{(-0.05946)(24)} \approx 0.2400$, then

$$a_i = 5r\left(\frac{1 - r^i}{1 - r}\right).$$

See me if you would like to see the derivation, but try it yourself first! (It is not difficult, but you will need the formula for the sum of a finite geometric series.) We can find a^* by taking the limit as $i \to \infty$. Since $0 < r < 1$, $r^i \to 0$ as $i \to \infty$; thus

$$a^* = \lim_{i\to\infty} a_i = \frac{5r}{1 - r} \approx \frac{1.2}{0.76} \approx 1.579.$$