Homework Assignment 2

Due *Friday, September 20.*

1. On the last page of this assignment are five differential equations for $u(t)$, and five slope fields. Match the equation with the slope field.

2. In each of the slope fields in Problem 1, sketch the function $u(t)$ for which $u(0) = 1$.

3. Consider an object falling through a viscous medium. According to Newton's second law of motion, the motion of an object is governed by the equation $F = ma$, where F is the total force acting on the object, m is the mass of the object, and a is the acceleration of the object. If we let $y(t)$ be the height of the object, then $v(t) = \frac{dy}{dt}$ is the velocity, and $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ is the acceleration. Note that with this choice of y, a positive velocity ($v > 0$) means the object is moving *up*.

   The forces acting on the object are gravity (a constant force given by mg, where g is the acceleration due to gravity), and friction. The frictional force depends on how fast the object is moving, and is always directed in the opposite direction of the motion. Let $F_d(v)$ be the force due to friction (the d subscript refers to *drag*).

   (a) Write down the differential equation satisfied by v, the velocity of the object. Assume that $g > 0$ (so you must be sure to put the correct sign in the equation), and that when $v > 0$, $F_d(v) < 0$.
   If the object is released from height y_0 when $t = 0$, what is the initial condition for the differential equation?

   (b) Consider *no friction*, so $F_d(v) = 0$. Solve the initial value problem for $v(t)$. What happens to v at t increases? That is, what is the long-term behavior of v?

   (c) Suppose the friction force is proportional to the velocity. That is, let $F_d(v) = -C_d v$, where $C_d > 0$ is a constant (the *drag coefficient*). Solve the initial value problem. What happens to v as t increases?

   (d) Now suppose that the friction force is proportional to the square of the velocity: let $F_d(v) = -C_d v|v|$. Solve the new problem, and describe the long-term behavior.
   Note that $v|v| = \begin{cases} v^2 & v \geq 0 \\ -v^2 & v \leq 0 \end{cases}$. I've written it this way to ensure that the friction force always opposes the velocity.
   Also, observe that for any non-zero constant a,

   $$\frac{1}{v^2 - a^2} = \frac{1}{(v+a)(v-a)} = \frac{-1}{2a(v+a)} + \frac{1}{2a(v-a)}.$$

   This is an example of a *partial fraction* expansion. It can be useful when you try to integrate the separated equation.

   (e) If y is measured in meters, t in seconds, and mass in kilograms, what must the units of C_d be in the previous two questions? (Note: the units are not the same in the two cases.)

4. The logistic equation

   $$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

   is a first order autonomous differential equation, so it is separable. Find the general solution to this differential equation.

Note: To integrate the separated equation, the following may be helpful:

$$\frac{1}{\left(1-\frac{P}{N}\right)P} = \frac{1}{N-P} + \frac{1}{P}.$$

This is another example of a partial fraction expansion.

5. We consider again the clearance of a drug from a patient's blood. Suppose $a(t)$ is the amount of a drug in the bloodstream, measured in milligrams. We assume that the drug "clears" at a rate that is proportional to the amount present, and we take the proportionality constant to be 0.2 per hour.

   Suppose that the patient is being given the drug intravenously. The drug is dissolved in a solution with a concentration of 1.4 milligrams per liter, and this solution enters the bloodstream continuously at the rate of 0.1 liters per hour.[1] The body regulates the volume of the blood so that the total volume of blood remains 5 liters. Assume that when the patient first starts receiving the drug, there is no drug already in the blood.

   (a) Give the initial value problem for $a(t)$ that models this situation.

   (b) Solve the initial value problem.

   (c) What is the amount of the drug in the bloodstream after one hour? After one day? After a "long time"? (In other words, what is $\lim_{t\to\infty} a(t)$?)

Text Problems:

- Section 1.4/ 2, 5, 6, 10, 11

[1] This is not meant to be a realistic flow rate for the intravenous administration of a drug.

Differential equations for Question 1:

1. $\frac{du}{dt} = \frac{1}{2}$ Slope Field: ____________

2. $\frac{du}{dt} = t - 1$ Slope Field: ____________

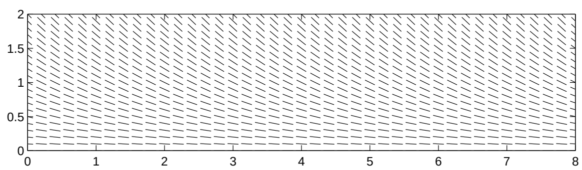
3. $\frac{du}{dt} = -\frac{1}{2}u$ Slope Field: ____________

4. $\frac{du}{dt} = \frac{1}{2}u$ Slope Field: ____________

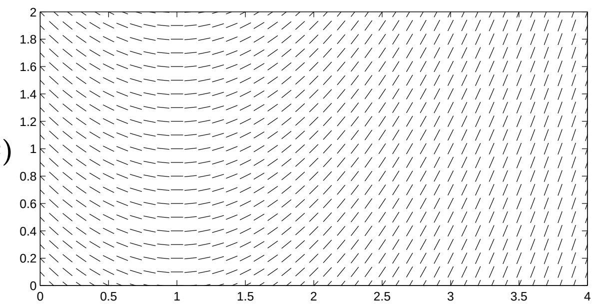
5. $\frac{du}{dt} = -\frac{1}{2}u\cos(t)$ Slope Field: ____________

Slope Fields for Questions 1 and 2:

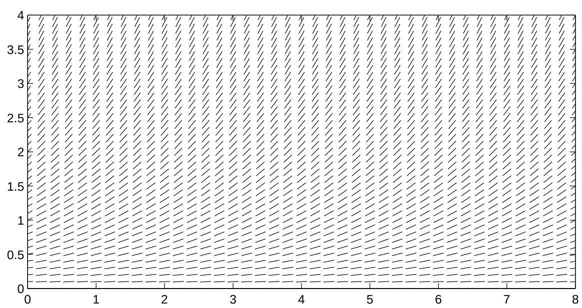
(1)



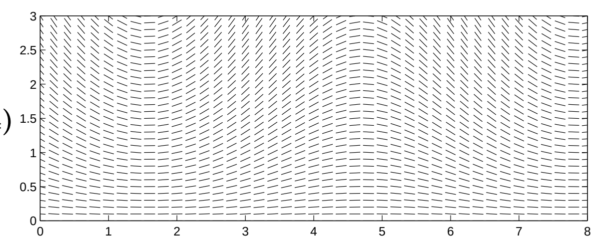
(2)



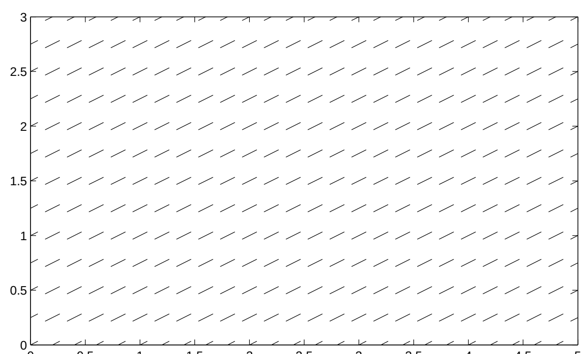
(3)



(4)



(5)



In all of the above plots, the horizontal axis is t, and the vertical axis is u.