Homework 2 Solutions

1 & 2. See the last page.

3(a). Newton's second law of motion says that $ma = F$, and we know $a = \frac{dv}{dt}$, so we have $m\frac{dv}{dt} = F$. One part of the force is gravity, mg. However, we are using the convention that "up" is positive, and gravity creates an acceleration downward, so the gravitational force is really $-mg$. In part (a), we are told that there is a frictional force $F_d(v)$ that depends on v. We know that this force should act in the direction opposite to the direction of motion, but we are told that $F_d(v) < 0$ when $v > 0$, so that is accounted for in F_d. So the total force is $F = -mg + F_d(v)$. Thus Newton's second law gives us

$$m\frac{dv}{dt} = -mg + F_d(v),$$

or

$$\frac{dv}{dt} = -g + \frac{F_d(v)}{m}.$$

This is a first order differential equation for v, the velocity of the object.

If the object is released from height y_0 at $t = 0$, the *velocity* at that instant is zero. Therefore the initial condition is

$$v(0) = 0.$$

3(b). In this case, $F_d(v) = 0$, and the equation is

$$\frac{dv}{dt} = -g.$$

We can integrate this to obtain

$$v = -gt + C,$$

and the initial condition $v(0) = 0$ gives us $C = 0$, so

$$v(t) = -gt.$$

As t increases, the velocity decreases linearly with t.

3(c). If $F_d(v) = -C_d v$, the equation for v is

$$\frac{dv}{dt} = -g - \frac{C_d}{m}\,v.$$

This equation for v is separable, so we can following the usual steps to solve it:

Separate (assuming $v \neq -gm/C_d$):

$$\frac{dv}{g + (C_d/m)v} = -dt.$$

Integrate and solve for $v(t)$. I'll do this one in detail once more (in the following, $\exp(x) \equiv e^x$):

$$\begin{aligned}
\frac{m}{C_d} \ln|g + (C_d/m)v| &= -t + k_1, \\
\ln|g + (C_d/m)v| &= -\frac{C_d}{m}t + k_2 \quad \text{(where } k_2 = (C_d/m)k_1\text{)}, \\
|g + (C_d/m)v| &= \exp\left(-\frac{C_d}{m}t + k_2\right) \\
&= k_3 \exp\left(-\frac{C_d}{m}t\right) \quad \text{(where } k_3 = e^{k_2}\text{)},
\end{aligned}$$

Note that k_3 must be a *positive* constant, because it comes from exponentiating k_2. Let's get rid of the absolute value. We have two cases: $g + (C_d/m)v > 0$, or $g + (C_d/m)v < 0$.

If $g + (C_d/m)v > 0$, then $|g + (C_d/m)v| = g + (C_d/m)v$, and the last equation becomes

$$g + (C_d/m)v = k_3 \exp\left(-\frac{C_d}{m}t\right)$$

If $g + (C_d/m)v < 0$, then $|g + (C_d/m)v| = -(g + (C_d/m)v)$, so we have

$$g + (C_d/m)v = -k_3 \exp\left(-\frac{C_d}{m}t\right)$$

The only difference between these two cases is the minus sign. Now, k_3 is a positive constant, so we can combine these two cases into one formula

$$g + (C_d/m)v = k_4 \exp\left(-\frac{C_d}{m}t\right)$$

where k_4 is an *arbitrary* nonzero constant. And then we observe that if $k_4 = 0$, we have $g + (C_d/m)v = 0$, or $v = -mg/C_d$, which is the equilibrium solution. Thus we can say that k_4 is an arbitrary constant (including the possibility that $k_4 = 0$).

Finally, we solve for v:

$$v(t) = -\frac{gm}{C_d} + k \exp\left(-\frac{C_d}{m}t\right),$$

where k is an arbitrary constant. (k is just $(C_d/m)k_4$, and since k_4 is arbitrary, so is k.)

We want the solution where $v(0) = 0$, so

$$v(0) = -\frac{gm}{C_d} + k = 0, \quad \text{hence} \quad k = \frac{gm}{C_d}.$$

Thus the solution to the initial value problem in this case is

$$v(t) = -\frac{gm}{C_d} + \frac{gm}{C_d} \exp\left(-\frac{C_d}{m}t\right) = -\frac{gm}{C_d}\left(1 - \exp\left(-\frac{C_d}{m}t\right)\right).$$

As t increases, $\exp\left(-\frac{C_d}{m}t\right)$ approaches zero, so the velocity approaches $-\frac{gm}{C_d}$. (This is the *terminal velocity.*)

3(d). We now assume that $F_d(v) = -C_d v|v|$. Before proceeding, let's use a little intuition to make life easier. If we drop the object, we expect it to fall; the velocity will be zero initially, and then it will become negative. We expect it to *stay* negative for $t > 0$. Therefore, for the solution that we are considering, we have $|v| = -v$, and $F_d(v) = C_d v^2$. The differential equation is then

$$\frac{dv}{dt} = -g + \frac{C_d v^2}{m}.$$

This equation is separable, and we solve it the usual way. That is, we write

$$\frac{dv}{-g + \frac{C_d v^2}{m}} = dt,$$

and integrate. While it is not necessary, I'll first make the coefficient of v^2 one:

$$\frac{dv}{-\frac{mg}{C_d} + v^2} = \frac{C_d}{m} dt,$$

For convenience, I'll define the parameter $a = \sqrt{\frac{mg}{C_d}}$. (We can do this because m, g, and C_d are all positive. Note that this is *not* the same a as the acceleration in Newton's Law.) We then have

$$\frac{dv}{v^2 - a^2} = \frac{C_d}{m} dt, \tag{1}$$

To integrate the expression on the left, we will use the partial fraction expansion

$$\frac{1}{v^2 - a^2} = \frac{-1}{2a(v+a)} + \frac{1}{2a(v-a)}.$$

Then

$$\begin{aligned}
\int \frac{dv}{v^2 - a^2} &= \int \frac{-dv}{2a(v+a)} + \int \frac{dv}{2a(v-a)} \\
&= -\frac{1}{2a} \ln|v+a| + \frac{1}{2a} \ln|v-a| \\
&= \frac{1}{2a} \left(\ln|v-a| - \ln|v+a| \right) \\
&= \frac{1}{2a} \ln\left| \frac{v-a}{v+a} \right|.
\end{aligned}$$

Thus, the result of integrating both sides of (1) is

$$\frac{1}{2a} \ln\left| \frac{v-a}{v+a} \right| = \frac{C_d}{m} t + K_1.$$

This gives us

$$\left| \frac{v-a}{v+a} \right| = \exp\left(\frac{2aC_d}{m} t + 2aK_1 \right) = K_2 \exp\left(\frac{2aC_d}{m} t \right),$$

or

$$\frac{v-a}{v+a} = K_3 \exp\left(\frac{2aC_d}{m} t \right).$$

where, as usual, we absorb the $\pm$ into the constant K_3. Now solve this for v:

$$\begin{aligned}
v - a &= K_3 \exp\left(\frac{2aC_d}{m} t \right) (v + a), \\
\left(1 - K_3 \exp\left(\frac{2aC_d}{m} t \right) \right) v &= a + K_3 \exp\left(\frac{2aC_d}{m} t \right) a = a \left(1 + K_3 \exp\left(\frac{2aC_d}{m} t \right) \right) \\
v &= \frac{a \left(1 + K_3 \exp\left(\frac{2aC_d}{m} t \right) \right)}{1 - K_3 \exp\left(\frac{2aC_d}{m} t \right)}
\end{aligned}$$

Recall that $a = \sqrt{\frac{mg}{C_d}}$, so we have

$$v = \sqrt{\frac{mg}{C_d}} \left(\frac{1 + K_3 \exp\left(2\sqrt{\frac{gC_d}{m}}\, t\right)}{1 - K_3 \exp\left(2\sqrt{\frac{gC_d}{m}}\, t\right)} \right)$$

Now we find the solution for which $v(0) = 0$:

$$v(0) = \sqrt{\frac{mg}{C_d}} \left(\frac{1 + K_3}{1 - K_3} \right) = 0 \implies K_3 = -1.$$

So the solution to the initial value problem is

$$v(t) = \sqrt{\frac{mg}{C_d}} \left(\frac{1 - \exp\left(2\sqrt{\frac{gC_d}{m}}\, t\right)}{1 + \exp\left(2\sqrt{\frac{gC_d}{m}}\, t\right)} \right)$$

As t increases, the exponentials in this formula become large (much larger than 1), and the expression in parentheses approaches -1. Thus, as t increases, $v(t)$ approaches $-\sqrt{\frac{mg}{C_d}}$. Again we see that there is a terminal velocity.

3(e). In (c), we have the equation $\frac{dv}{dt} = -g - (C_d/m)v$, so the term $(C_d/m)v$ must have the same units as $\frac{dv}{dt}$, which is an acceleration, with units m/sec^2. v has units of m/sec, and m has units of kg, so C_d must have units of **kg/sec** for $(C_d/m)v$ to have units of m/sec^2.

In (d), the term $(C_d/m)v^2$ must have units of acceleration, so in this case, the units of C_d must be **kg/m**.

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4. The logistic equation is

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

The equilibria are $P(t) = 0$ and $P(t) = N$. Now assume $P \neq 0$ and $P \neq N$. We separate to obtain:

$$\frac{dp}{\left(1 - \frac{P}{N}\right)P} = k\,dt.$$

To integrate the left side, we use the partial fraction expansion:

$$\int \frac{dp}{\left(1 - \frac{P}{N}\right)P} = \int \left(\frac{1}{N-P} + \frac{1}{P}\right) dp = -\ln|N-P| + \ln|P| = \ln\left|\frac{P}{N-P}\right|.$$

(I haven't included the constant of integration; this will be included in the constant on the right side.) So after integrating we have

$$\ln\left|\frac{P}{N-P}\right| = kt + C_1.$$

Now solve for P:

$$\left|\frac{P}{N-P}\right| = e^{kt+C_1} = C_2 e^{kt},$$

$$\frac{P}{N-P} = C_3 e^{kt}$$

Note that C_2 is a *positive* constant, while C_3 is an arbitrary nonzero constant. As in previous problems, the $\pm C_2$ that arises from eliminating the absolute value is absorbed into C_3.

With a little algebra, we find solve for P:

$$P(t) = \frac{C_3 N e^{kt}}{1 + C_3 e^{kt}}. \tag{2}$$

We observe that setting $C_3 = 0$ gives us $P = 0$, which is one of the equilibrium solutions, so in fact we can allow C_3 to be an arbitrary constant, including $C_3 = 0$. However, there is no value of C_3 that will give us the other equilibrium solution $P = N$, so the general solution is given by (2) or

$$P(t) = N.$$

A different (but equivalent) form of the solution is

$$P = \frac{N}{1 + C_4 e^{-kt}} \quad \text{or} \quad P(t) = 0,$$

where C_4 is an arbitrary constant.

5(a). The amount of drug in the bloodstream at time t (in hours) is $a(t)$ milligrams. There are two processes causing the amount to change:

1. The drug clears at a rate that is proportional to the amount present. We are told that the proportionality constant is 0.2 per hour, so the rate of change of $a(t)$ from the clearing of the drug is $-0.2a(t)$.
2. The drug is being added intravenously. The concentration of the drug in the solution is 1.4 mg/ℓ, and the flow rate is 0.1 ℓ/hour, so the rate of change of a is (1.4 mg/ℓ)(0.1 ℓ/hour) = 0.14 mg/hour.

Thus the net rate of change of $a(t)$ is $-0.2a + 0.14$. This gives us the differential equation

$$\frac{da}{dt} = -0.2a + 0.14.$$

We are told that there is initially no drug in the bloodstream, so the initial condition is

$$a(0) = 0.$$

The differential equation and the initial condition are the initial value problem.

5(b). The differential equation is separable; in fact, we solved a more general equation like this in class. We find the general solution to the differential equation to be

$$a(t) = 0.7 + Ce^{-0.2t}.$$

We use the initial condition to find that $C = -0.7$, so the solution to the initial value problem is

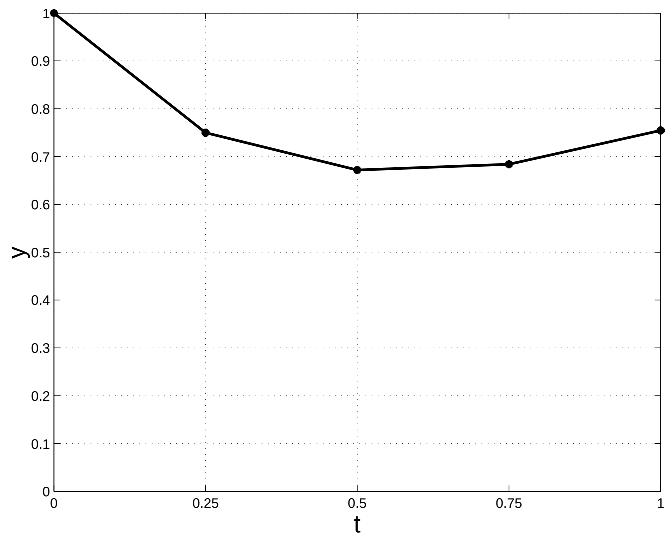
$$a(t) = 0.7 - 0.7e^{-0.2t}.$$

5(c).

- After one hour, the amount of drug in the bloodstream is $a(1) = 0.7 - 0.7e^{-0.2} = 0.1269$ mg.
- After one day, $a(24) = 0.6942$ mg.
- As $t \to \infty$, $e^{-0.2t} \to 0$, so $\lim_{t\to\infty} a(t) = 0.7$.

1.4/2. The following table and plot show the result of applying Euler's Method.

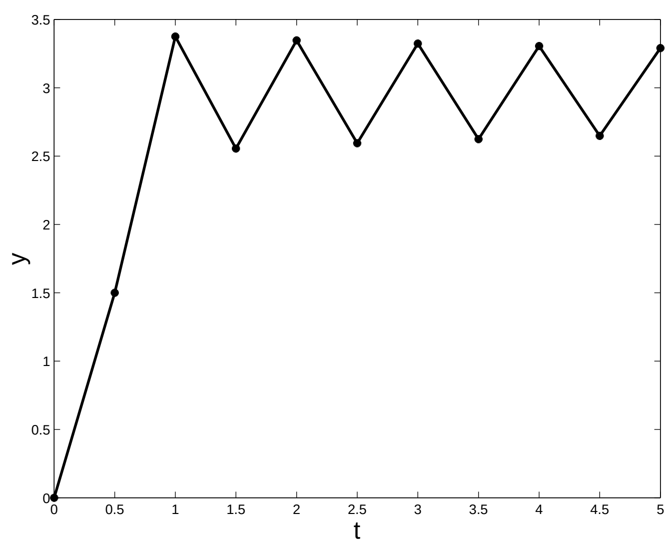
k	t_k	y_k	$f(t_k, y_k)$
0	0.00	1.0000	-1.0000
1	0.25	0.7500	-0.3125
2	0.50	0.6719	0.0486
3	0.75	0.6840	0.2821
4	1.00	0.7545	0.4307



1.4/5. (See the solution in the text.)

1.4/6.

k	t_k	y_k	$f(t_k, y_k)$
0	0	0	3.0000
1	0.5000	1.5000	3.7500
2	1.0000	3.3750	-1.6406
3	1.5000	2.5547	1.5829
4	2.0000	3.3462	-1.5045
5	2.5000	2.5939	1.4594
6	3.0000	3.3236	-1.3992
7	3.5000	2.6240	1.3626
8	4.0000	3.3053	-1.3144
9	4.5000	2.6481	1.2838
10	5.0000	3.2900	-1.2441

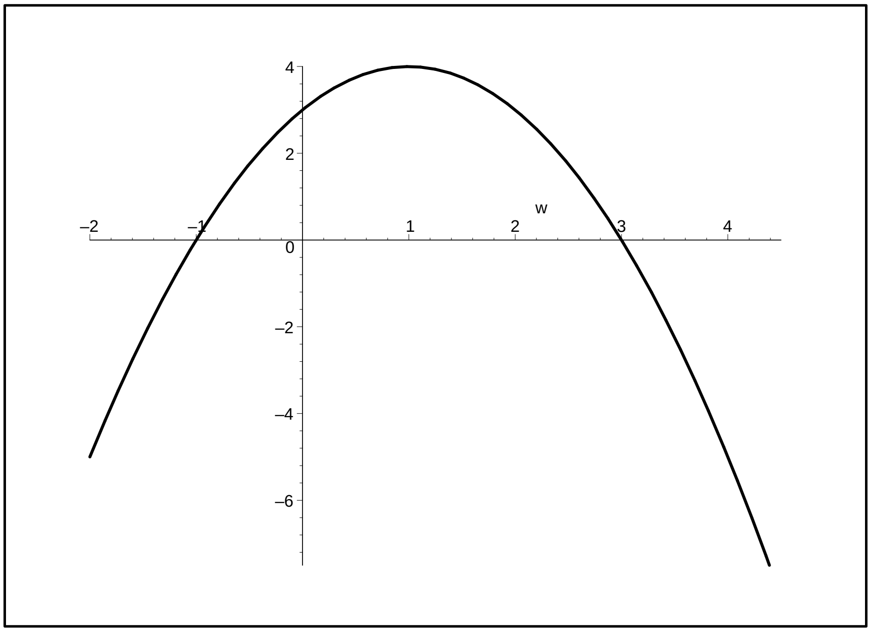


1.4/10. Euler's Method is not working very well in either case. For example, we know the equation has an equilibrium at $w = 3$. If $w(0) > 3$, we expect the solution to approach this equilibrium as $t \to \infty$. But in Exercise 5, the solution jumps from $w = 4$ to $w = -1$ in one step, and in Exercise 6, the numerical solution appears to be oscillating around $w = 3$. Neither of these behaviors is reasonable for this equation. Any solution that crossed $w = 3$ would have to have a slope of zero there; neither of these approximations shows this. (Looking ahead to Section 1.5, we could quote the Uniqueness Theorem, and point out that it would be *impossible* for a solution to cross $w = 3$.)

In both cases, the step size is too big. We need to use a smaller step size to get reasonable results.

1.4/11. Here is a plot of the right-hand side of the differential equation in Exercise 6 versus w:

$$\frac{dw}{dt} \text{ vs. } w$$



There are two equilibrium solutions, $w(t) = -1$ and $w(t) = 3$. For $w < -1$ or $w > 3$, $w(t)$ is decreasing, while for $-1 < w < 3$, $w(t)$ is increasing. Any solution $w(t)$ for which $-1 < w(0) < 3$ will increase monotonically toward 3. If a solution crossed $w = 3$ (which, as we will see in Section 1.5, is not possible for this equation), it would have to have a slope of zero there.

The numerical solution computed in Exercise 6 oscillates around the value $w = 3$, which our qualitative analysis shows is impossible behavior for a solution to the differential equation.