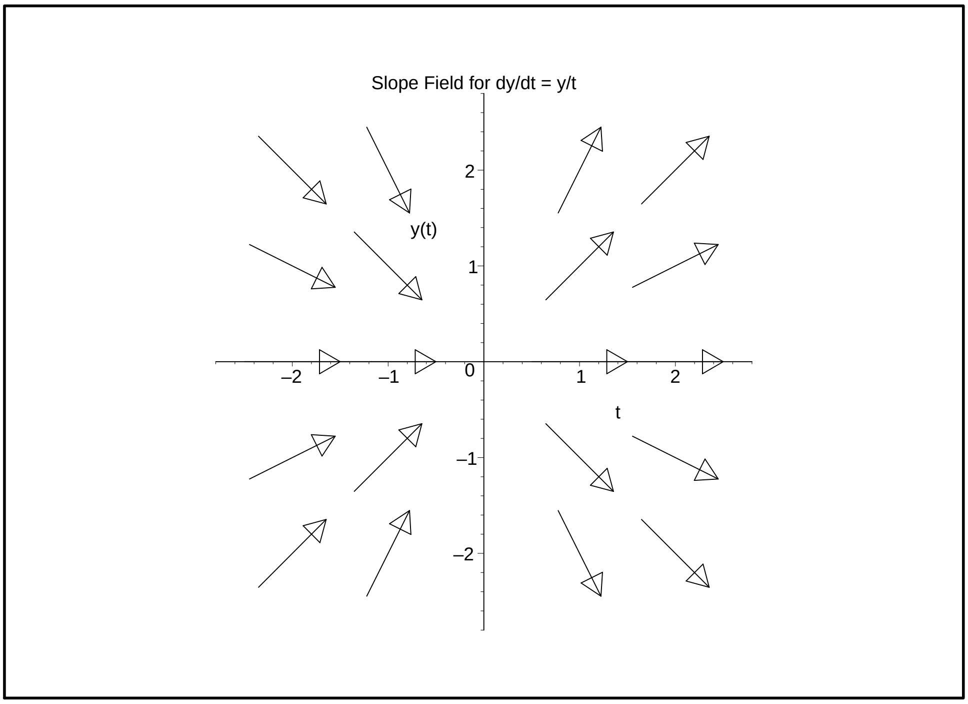
Homework 3 Solutions (Partial)

1(a). The right-hand side of the differential equation is $f(t, y) = y/t$, and this function is *not continuous* at $t = 0$; in fact, it is not defined there. Also, $\frac{\partial f}{\partial y} = \frac{1}{t}$, which is also not continuous at $t = 0$. The functions f and $\frac{\partial f}{\partial y}$ are continuous everywhere else in the ty plane, so the Existence and Uniqueness Theorems (as given in the text and in class) apply in any rectangular region in the ty plane that does not contain $t = 0$.

1(b). Here is the slope field:



All the arrows appear to point directly into or away from the origin. This suggests that the solutions might be straight lines through the origin. Let's see if this works. If $y = kt$ for some constant k, then $\frac{dy}{dt} = k$, and $\frac{y}{t} = k$, so $y = kt$ is, in fact, a solution to the differential equation. (This is an example of the "guess and check" method of solving a differential equation.)

1(c). We follow the usual steps for a separable equation. We assume that $y \neq 0$ and $t \neq 0$. Then

$$\frac{dy}{dt} = \frac{y}{t} \implies \frac{dy}{y} = \frac{dt}{t} \implies \ln|y| = \ln|t| + C_1 \implies |y| = e^{\ln|t|+C_1} = C_2 e^{\ln|t|} = C_2|t|.$$

So we have

$$|y| = C_2|t|,$$

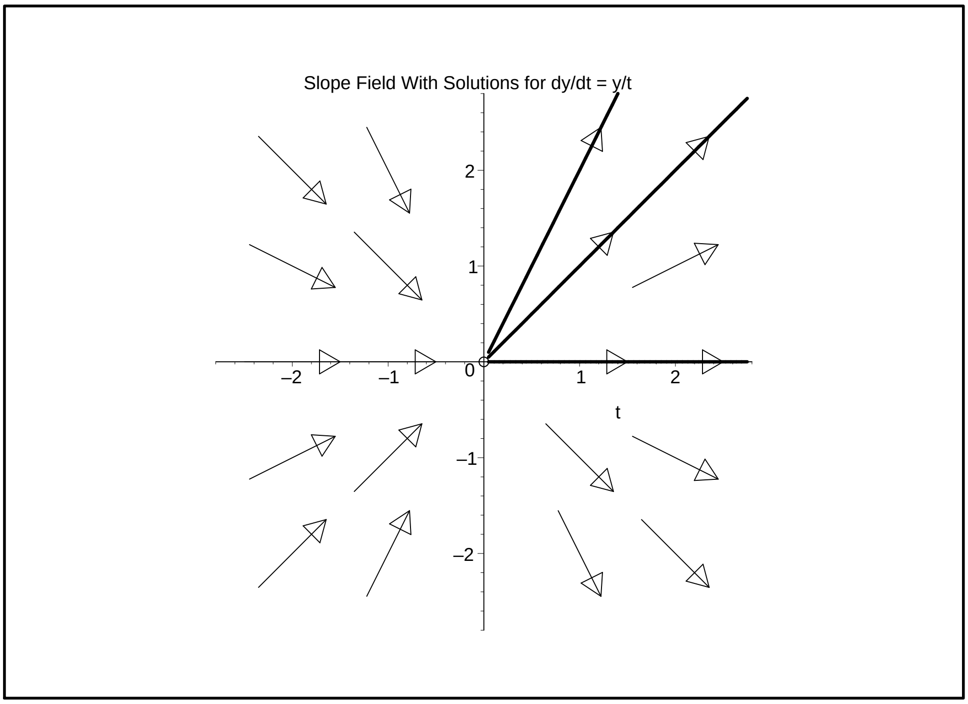
where C_2 is some *positive* constant. If $y > 0$ and $t > 0$, *or* if $y < 0$ and $t < 0$, this equation becomes $y = C_2 t$ (a line with positive slope). If $y < 0$ and $t > 0$, *or* if $y > 0$ and $t < 0$, this equation becomes $y = -C_2 t$ (a line with negative slope). Since $y = 0$ is an equilibrium solution (for $t > 0$ or $t < 0$), we can express all the solutions with one formula:

$$y = C_3 t,$$

where C_3 is an arbitrary constant. (This is what we predicted above.)

- If $y(1) = 0$, then $C_3 = 0$, and the solution is $y(t) = 0$.
- If $y(1) = 1$, then $C_3 = 1$, and the solution is $y(t) = t$.
- If $y(1) = 2$, then $C_3 = 2$, and the solution is $y(t) = 2t$.

Here is the slope field with the solutions $y(t) = 0$, $y(t) = t$, and $y(t) = 2t$ included:



1(d). We found that the general solution is $y(t) = C_3 t$, and therefore $y(0) = 0$. This does *not* contradict the Uniqueness Theorem, because the Uniqueness Theorem does not apply when $t = 0$ for this differential equation.

Note that in the above plot, I have only plotted the solution for $t > 0$. Technically, since we can not evaluate the right side of the differential equation at $t = 0$, we should not include $t = 0$ in our solutions. Then, for any solution to the initial value problem $y(t_0) = y_0$ with $t_0 > 0$, the "biggest" interval of t for which that solution can be a continuous solution to the differential equation is $0 < t < \infty$.

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2. The differential equation is a first order autonomous differential equation, so it is separable. First we observe that $y(t) = 0$ is an equilibrium solution. Now assume that $y \neq 0$, and following the usual steps:
Separate:

$$y^{-\frac{4}{5}} dy = 5dt$$

Integrate:

$$5y^{\frac{1}{5}} = 5t + C_1$$

Solve for y:

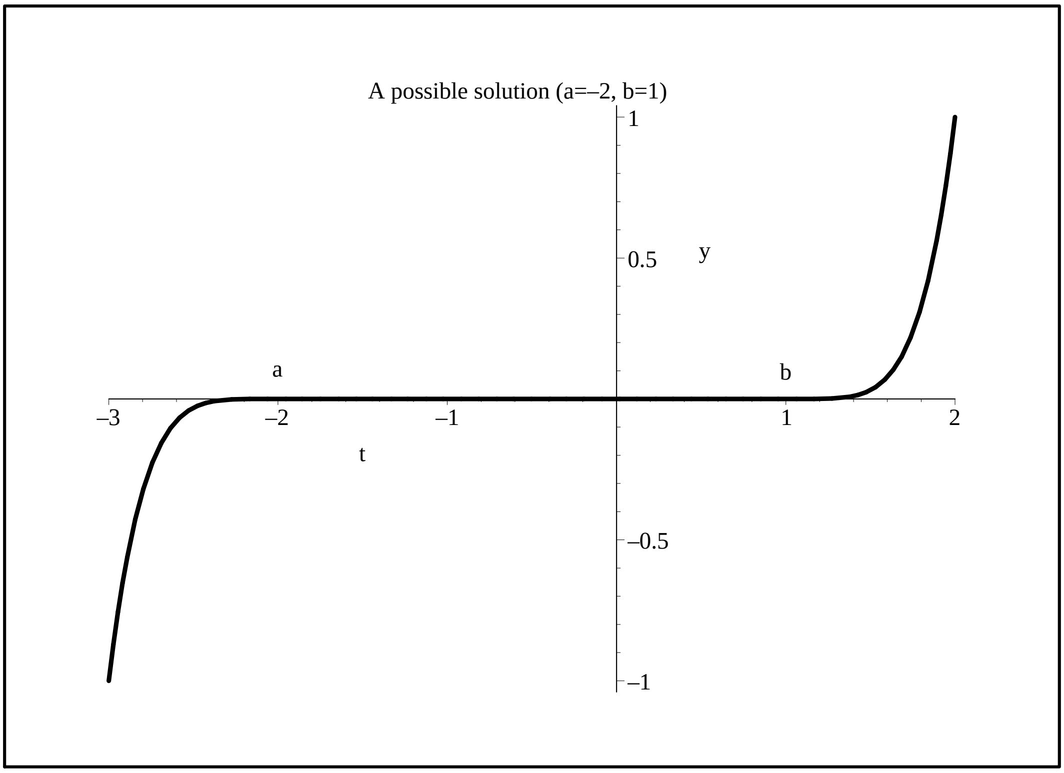
$$y = (t + C_2)^5, \quad \text{where} \quad C_2 = C_1/5.$$

The initial condition $y(0) = 0$ gives $y(0) = (C_2)^5 = 0$, so $C_2 = 0$.

So we now have two solutions that satisfy the initial condition $y(0) = 0$: $y(t) = 0$ and $y(t) = t^5$. We can use these two solutions to construct more solutions. Let a and b be two constants such that $a \leq 0$ and $b \geq 0$. Then any function of the form

$$y(t) = \begin{cases} (y-a)^5 & t < a \\ 0 & a \leq t \leq b \\ (y-b)^5 & t > b \end{cases}$$

is also a solution to the initial value problem. The following plot shows an example where $a = -2$ and $b = 1$.



Text Problems:

1.5/11(a). Recall from calculus that if the derivative exists at a local maximum, then it must be zero there. That is, $\frac{dy_1}{dt} = 0$ at $t = t_0$. But $y_1(t)$ is a solution to the differential equation, so $\frac{dy_1}{dt} = f(y_1(t))$. Thus, at $t = t_0$, $f(y_1(t_0)) = 0$; and since $y_0 = y_1(t_0)$, we have $f(y_0) = 0$.

1.5/11(b). The tangent line segments along $y = y_0$ are horizontal (i.e. have zero slope) for all values of t, because $f(y_0) = 0$.

1.5/11(c). We have $\frac{dy_2}{dt} = \frac{d}{dt}(y_0) = 0$, and we also have $f(y_2(t)) = f(y_0) = 0$, so the function $y_2(t) = y_0$ is a solution to the differential equation; it is a constant function, so it is an equilibrium solution.

1.5/11(d). Apparently, we have two solutions that go through the point (t_0, y_0). However, we are told that $f(y)$ is continuously differentiable. This means that $\frac{\partial f}{\partial y}$ is continuous, and this means that the conditions of the uniqueness theorem are satisfied. Thus there can only be one solution through this point, so $y_1(t) = y_2(t)$. Thus, $y_1(t) = y_0$ for all t.

1.5/11(e). The same arguments apply to a function that has a local minimum at some time t_0, since the derivative of $y(t)$ is zero there.

Conclusion of 1.5/11: This problem has shown that the solution to an *autonomous* first order differential equation can not have a strict local maximum or a strict local minimum. This means that a solution either increases monotonically, decreases monotonically, or is an equilibrium.

1.5/14(a). We solve the equation in the usual way. We assume that $y \neq 0$; then
$\frac{dy}{dt} = y^3 \implies \frac{dy}{y^3} = dt \implies -y^{-2}/2 = t + C_1 \implies y^{-2} = -2t + C_2 \implies y = \frac{1}{\pm\sqrt{-2t+C_2}}$
The $\pm$ tells us that for each value of C_2, there are two possible solutions. We want $y(0) = 1 > 0$, so we must use the positive root. Then $y(0) = 1 \implies C_2 = 1$. Thus the solution to the initial value problem is

$$y(t) = \frac{1}{\sqrt{-2t+1}}.$$

1.5/14(b). This solution is defined as long as $-2t + 1 > 0$, so the domain of definition is $t < \frac{1}{2}$.

1.5/14(c). Consider $t \to \left(\frac{1}{2}\right)^-$. We see that $\lim_{t \to \left(\frac{1}{2}\right)^-} y(t) = \infty$; the solution "blows up" at $t = 1/2$.

Also, as $t \to -\infty$, $y(t) \to 0$.

1.5/17(a). We solve the differential equation in the usual way. We assume that $y \neq 2$; then

$$\frac{dy}{dt} = \frac{t}{y-2} \implies (y-2)dy = tdt \implies y^2/2 - 2y = t^2/2 + C_1 \implies y^2 - 4y - t^2 + C_2 = 0.$$

We are given $y(-1) = 0$, so $(0)^2 - 4(0) - (-1)^2 + C_2 = 0 \implies C_2 = 1$. We continue to solve for y:

$$y^2 - 4y - t^2 + 1 = 0 \implies y = \frac{4 \pm \sqrt{16 + 4t^2 - 4}}{2} \implies y = 2 \pm \sqrt{t^2+3}.$$

In order to satisfy the initial condition $y(-1) = 0$, we must use the *negative* square root, so the solution to the initial value problem is

$$y(t) = 2 - \sqrt{t^2+3}.$$

1.5/17(b). Since $t^2 + 3 > 0$ for all t, and $y(t) < 2$ for all t, the solution is defined for all t.

1.5/17(c). In this case, since the solution is defined for all t, the "limits of its domain" means $t \to \pm\infty$, and

$$\lim_{t\to\infty} y(t) = -\infty \quad \text{and} \quad \lim_{t\to-\infty} y(t) = -\infty.$$

1.6/41(a). This is a straight-forward application of the Intermediate Value Theorem (IVT) for continuous functions. We are told that f is continuous, $f(-10) > 0$ and $f(10) < 0$, so by the IVT, there is a number c between -10 and 10 such that $f(c) = 0$. This means that $y(t) = c$ is an equilibrium solution.

1.6/41(b). Because $y = 1$ is a *source*, there must be an interval to the right of $y = 1$, say $1 < y \le 1 + \epsilon$ (where $\epsilon > 0$), on which $f(y) > 0$, and an interval to the left of $y = 1$, say $1 - \delta \le y < 1$ (where $\delta > 0$) on which $f(y) < 0$.

Now, since $f(1+\epsilon) > 0$, and $f(10) < 0$, the IVT says there must be a number c between $1 + \epsilon$ and 10 such that $f(c) = 0$. There may, in fact, be several points where f is zero, but since there are only finitely many equilibria, and f must reach a negative value when $y = 10$, f must *strictly decrease* through one of these points, say c_1. Then the equilibrium $y = c_1$ is a sink.

Similarly, f must be zero at one or more points between -10 and $1 - \delta$, and at one of these points, say $y = c_2$, f must be strictly decreasing, so $y = c_2$ is also a sink.

1.6/43. First, a reminder: Taylor's Theorem says that (for a smooth enough function f), we may write

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + \frac{f''(y_0)}{2}(y - y_0)^2 + \frac{f'''(y_0)}{3!}(y - y_0)^3 + \frac{f^{iv}(z)}{4!}(y - y_0)^4,$$

where z is a number between y and y_0. We are assuming that y_0 is an equilibrium, so $f(y_0) = 0$, and we are left with

$$f(y) = f'(y_0)(y - y_0) + \frac{f''(y_0)}{2}(y - y_0)^2 + \frac{f'''(y_0)}{3!}(y - y_0)^3 + \frac{f^{(iv)}(z)}{4!}(y - y_0)^4.$$

Note that the terms on the right are polynomials in $(y - y_0)$; the first term is linear, the second is quadratic, etc. The behavior (or the shape) of the graph of f near y_0 is determined by the lowest order nonzero term in this formula. Suppose, for example, that $f'(y_0) \neq 0$. If we zoom in closer and closer to y_0, the graph will look more and more like a straight line, with slope $f'(y_0)$.

If $f'(y_0) = 0$, but $f''(y_0) \neq 0$, then as we zoom in closer and closer to y_0, the graph will look more and more like a parabola. Similarly, if both $f'(y_0)$ and $f''(y_0)$ are zero, but $f'''(y_0) \neq 0$, the graph will look like the cubic function $k(y - y_0)^3$ near y_0 (where $k = f'''(y_0)/3!$).

With this in mind, we can answer the questions.

(a) In this case, the graph of f looks like $\frac{f'''(y_0)}{3!}(y - y_0)^3$ when y is close to y_0, and since $f'''(y_0) > 0$, this implies that $f(y) > 0$ for y close to but greater than y_0, and $f(y) < 0$ for y close to but less than y_0. Therefore, y_0 must be a *source*.

(b) We make the same argument as above, but now $f'''(y_0) < 0$, so $f(y)$ has the opposite sign (near y_0) as the previous case, so y_0 is a *sink*.

(c) In this case, as we zoom in closer and closer to y_0, the function looks more and more like the parabola $\frac{f''(y_0)}{2}(y - y_0)^2$, and since $f''(y_0) > 0$, we have $f(y) > 0$ on either side of y_0. Thus y_0 is a *node*.