Math 308 – Differential Equations **Exam 1** **Spring 2002**

Name: ______________________________

You must show your work to receive full credit.

#	Points	Score
1	15	
2	21	
3	10	
4	10	
5	10	
6	12	
7	12	
8	10	
Total	100	

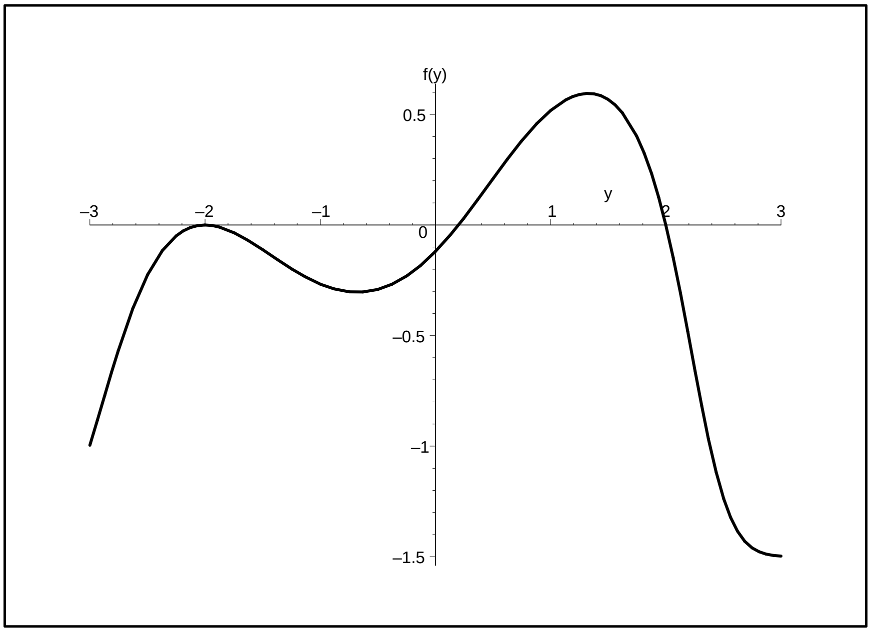
Reminder:

$$\int \frac{dx}{x^2+1} = \arctan(x) + C$$

1. Consider the differential equation

$$\frac{dy}{dt} = f(y),$$

where $f(y)$ is shown in the following plot:

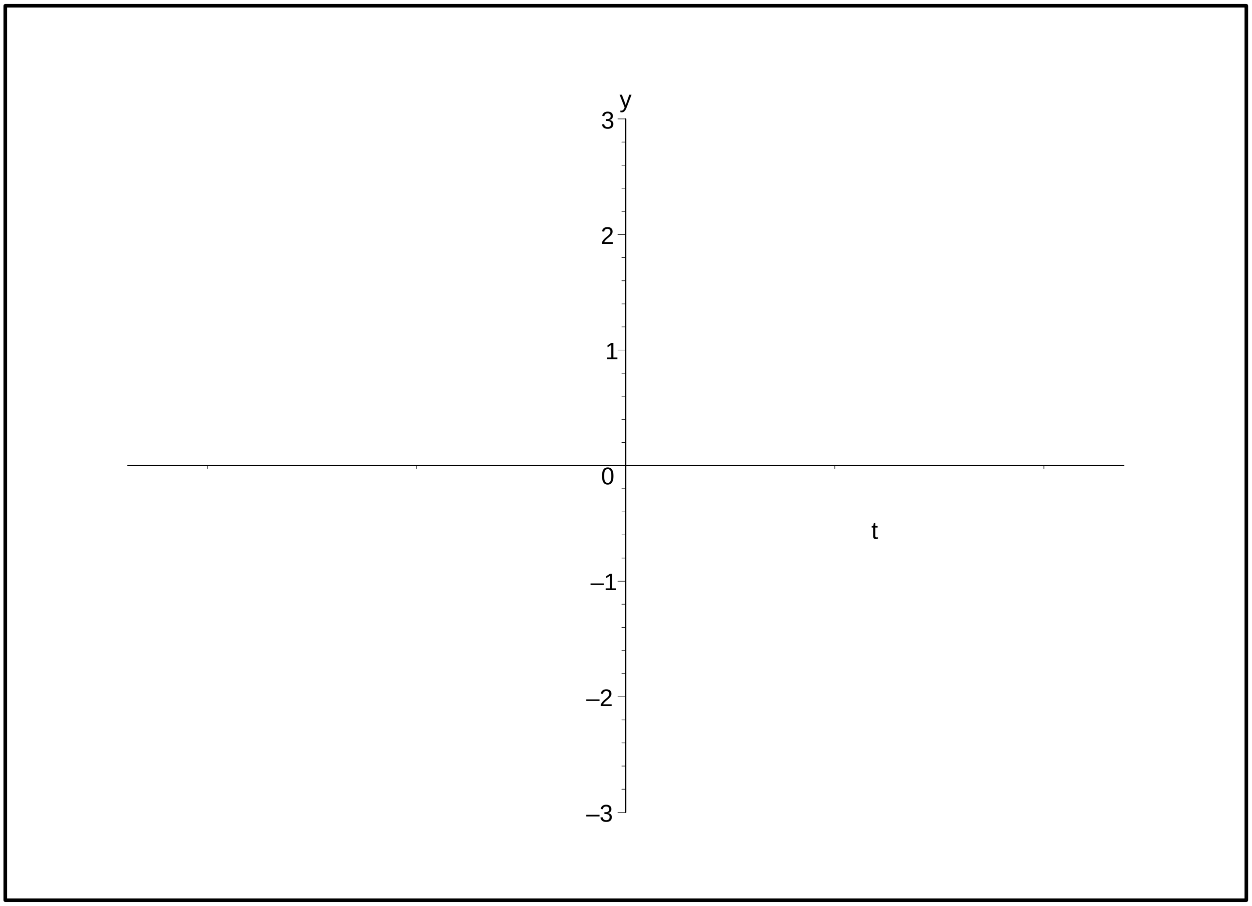


(The graph of f is tangent to the y axis at $y = -2$.)

(a) Sketch the phase line for this equation. (You may draw it in the above plot.)

(b) Identify the equilibria, and determine whether each equilibrium is a source, a sink, or a node.

(c) In the following set of axes, sketch the solutions $y(t)$ versus t for which $y(0) = -2.5$, $y(0) = -1.5$, $y(0) = -0.5$, $y(0) = 0.5$, $y(0) = 1.5$ and $y(0) = 2.5$, and sketch the equilibrium solutions. Include positive and negative values of t.



(d) Suppose $y(t)$ is the solution that satisfies $y(0) = 0$. Find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.

2. For each of the following differential equations, classify it as linear or not, autonomous or not, and separable or not. Then find the general solution.

   (a) $$\frac{dy}{dt} = 2y + 1$$

Linear:	YES / NO
Autonomous:	YES / NO
Separable:	YES / NO

   (b) $$\frac{dy}{dt} = 2y + e^{2t}$$

Linear:	YES / NO
Autonomous:	YES / NO
Separable:	YES / NO

(c) $\frac{dy}{dt} = ty^2 + 4t - y^2 - 4$

Linear:	YES / NO
Autonomous:	YES / NO
Separable:	YES / NO

3. Solve each of the following initial value problems. For each problem, state the interval of t for which the solution is defined.

(a) $\dfrac{dy}{dt} = y^2, \qquad y(0) = -2$

(b) $\dfrac{dy}{dt} = \dfrac{3t^2}{2y}, \qquad y(0) = -1$

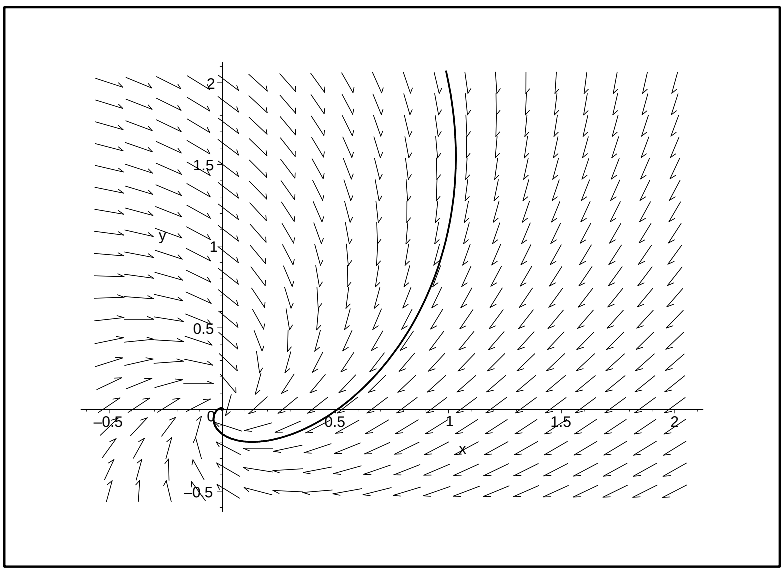
4. Find the general solution to

$$\frac{dy}{dt} = \frac{4}{t}y + t^4,$$

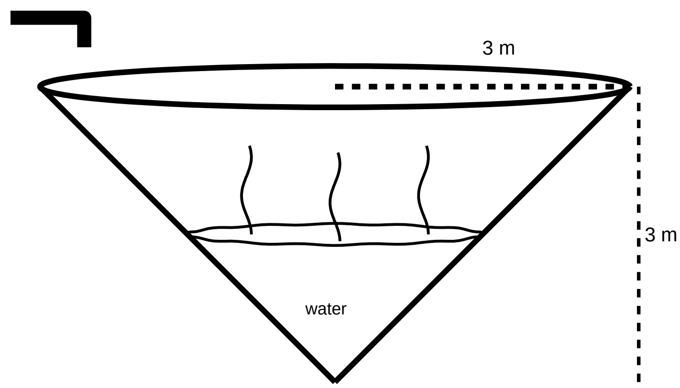
and show that for all solutions, $\lim_{t \to 0} y(t) = 0$. Explain why this does not contradict the Uniqueness Theorem.

5. (a) Suppose the initial value problem $\frac{dy}{dt} = f(y), \ \ y(0) = 1$, has the solution $y(t) = \sqrt{2t+1}$. What is the solution if the initial condition is changed to $y(0) = 2$? *Briefly* explain how you found your answer.

(b) The plot below shows a direction field for an autonomous system of two first order differential equations $dx/dt = f(x, y)$, $dy/dt = g(x, y)$. The solid curve corresponds to a solution $(x(t), y(t))$ to this system, with initial conditions $x(0) = 1$ and $y(0) = 2$. In the same plot, sketch the curve given by $(x(t-1), y(t-1))$.



6. Water flows into a cone-shaped tank (with height 3 meters and radius 3 meters) at a rate of 2 liters per minute. Water is lost by the evaporation of water at the surface. Assume that the rate at which water is lost by evaporation is proportional to the surface area, with a proportionality constant of 1/3 liters per square meter per minute. Suppose that the cone is initially empty.



   (a) Set up (but do not solve) the initial value problem for $v(t)$, the amount of water in the cone.

   (Potentially useful formulas: area of a circle of radius r is πr^2; volume of a cone with height h and base radius r is $\frac{\pi}{3} r^2 h$.)

(b) Use a qualitative analysis to determine the behavior of the solution to the initial value problem in (a). Include a rough sketch of $v(t)$ vs. t, and describe what happens to $v(t)$ as $t \to \infty$. Does the tank overflow?

7. Consider the following differential equation

$$\frac{dy}{dt} = (\mu - y)y,$$

where μ is a parameter. Describe how the number and type of equilibria depend on the parameter μ. Sketch the bifurcation diagram for this system, and determine the bifurcation value. Sketch three phase lines, one for μ less than, equal to, and greater than the bifurcation value.

8. Below are five different systems of two first order equations. The next page contains phase planes; within each phase plane is a solution corresponding to the initial condition $x(0) = 2$, $y(0) = 1$. The page after that contains plots of $x(t)$ and $y(t)$ versus t for the same solution shown in the corresponding phase plane.

Match the phase plane plots and the graphs of $x(t)$ and $y(t)$ to the systems below.

Note that in the graphs of $x(t)$ and $y(t)$ vs. t, only the interval $-2 \leq t \leq 2$ is shown. The t intervals for the curves shown in the phase planes overlap this interval, but they may include more or less at either end of the t interval.

(a) $$\frac{dx}{dt} = -x$$ $$\frac{dy}{dt} = y$$ **Phase Plane:** ____ **Graphs of $x(t)$ and $y(t)$:** ____

(b) $$\frac{dx}{dt} = -x$$ $$\frac{dy}{dt} = -2y$$ **Phase Plane:** ____ **Graphs of $x(t)$ and $y(t)$:** ____

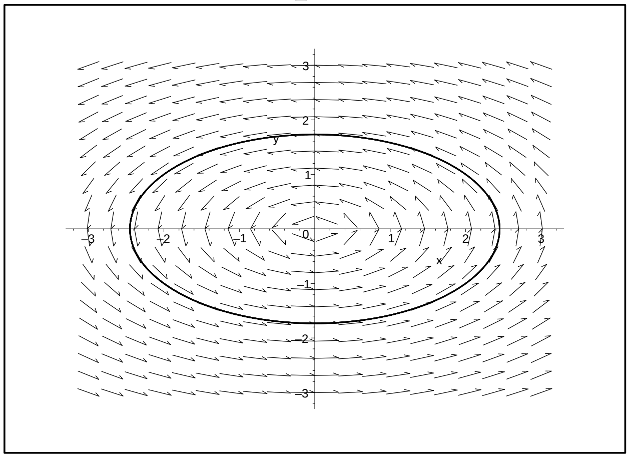
(c) $$\frac{dx}{dt} = -3y$$ $$\frac{dy}{dt} = \frac{3}{2}x$$ **Phase Plane:** ____ **Graphs of $x(t)$ and $y(t)$:** ____

(d) $$\frac{dx}{dt} = y$$ $$\frac{dy}{dt} = -\frac{1}{2}x - y$$ **Phase Plane:** ____ **Graphs of $x(t)$ and $y(t)$:** ____

(e) $$\frac{dx}{dt} = -x + xy$$ $$\frac{dy}{dt} = -y - \frac{1}{2}xy$$ **Phase Plane:** ____ **Graphs of $x(t)$ and $y(t)$:** ____

Phase Planes

1



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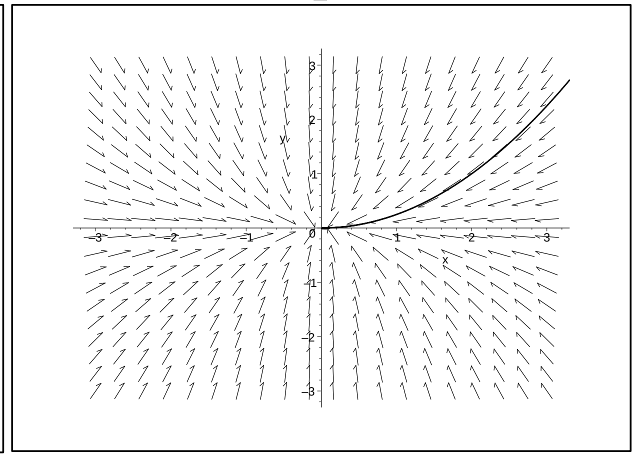

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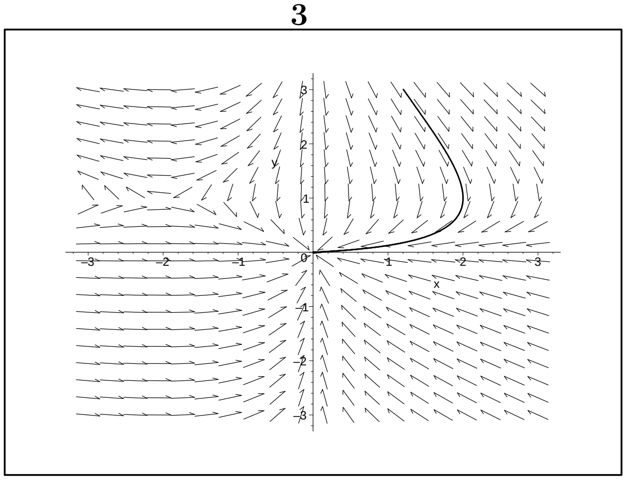

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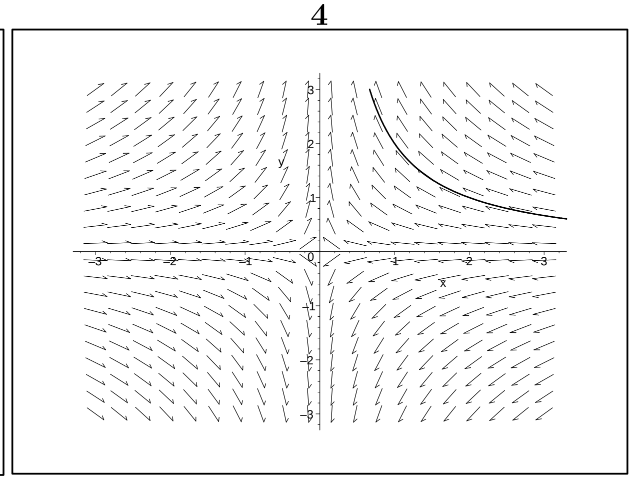

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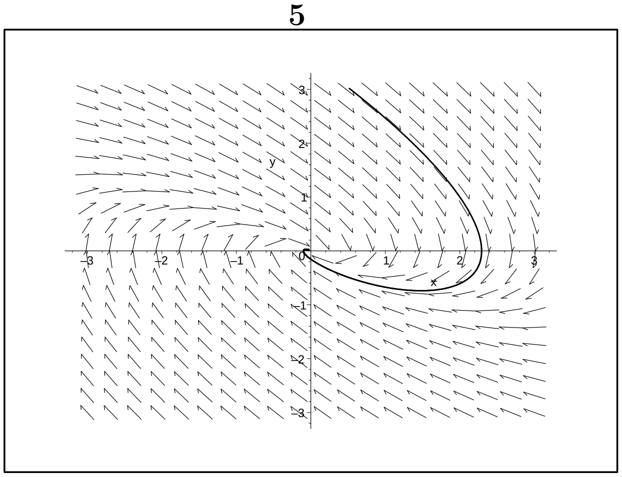
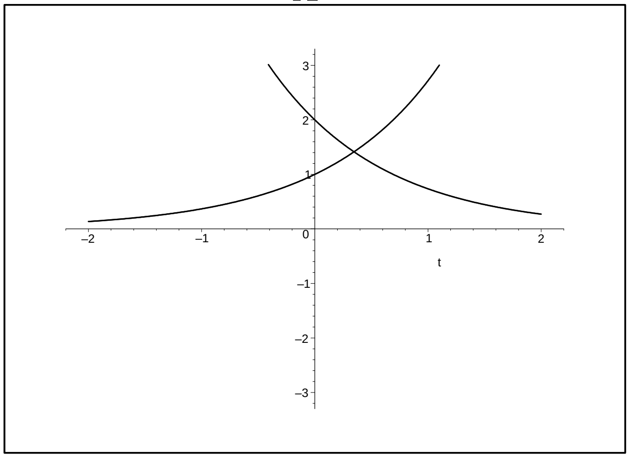
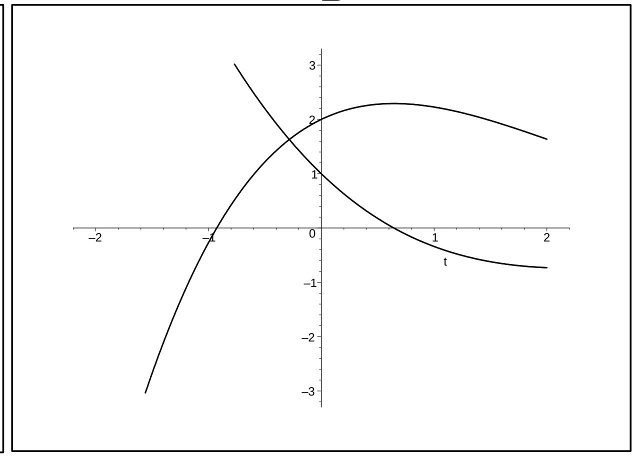


Graphs of $x(t)$ and $y(t)$ versus t

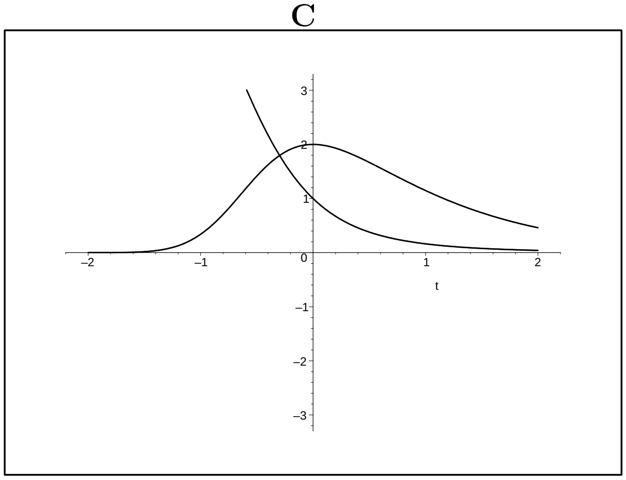
A



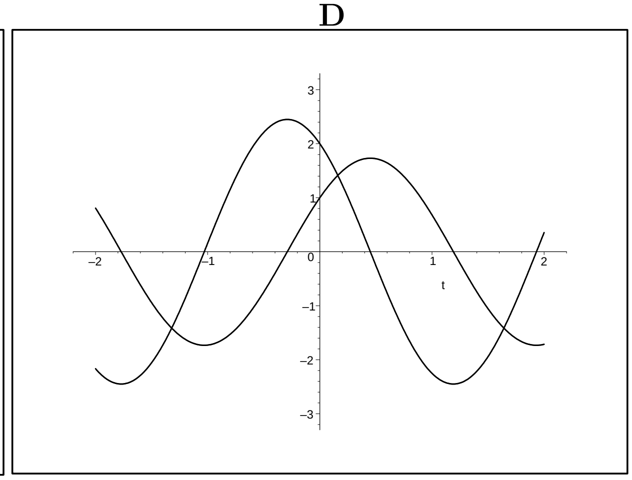
B



C



D



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