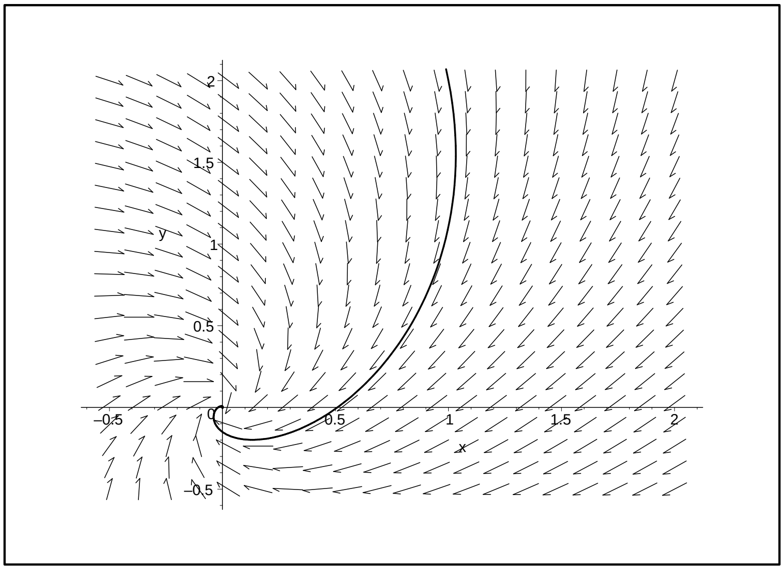
Math 308 – Differential Equations **Practice Exam 2** **Fall 2002**

1. The plot below shows a direction field for an autonomous system of two first order differential equations $dx/dt = f(x, y)$, $dy/dt = g(x, y)$. The solid curve corresponds to a solution $(x(t), y(t))$ to this system, with initial conditions $x(0) = 1$ and $y(0) = 2$. In the same plot, sketch the curve given by $(x(t-1), y(t-1))$.



2. The differential equations

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - \frac{dy}{dt} + x + 4y = 0$$

and

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 8y - 3x = 0$$

are a pair of coupled second order equations for $x(t)$ and $y(t)$. Convert these equations to a system of four first order equations, and write the system in matrix form. (*Hint:* Let $u = \frac{dx}{dt}$, and $v = \frac{dy}{dt}$.) Do not solve the equations!

3. Solve the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

and sketch $y(t)$ versus t.

4. Below are five different systems of two first order equations. The next page contains phase planes; within each phase plane is a solution corresponding to the initial condition $x(0) = 2$, $y(0) = 1$. The page after that contains plots of $x(t)$ and $y(t)$ versus t for the same solution shown in the corresponding phase plane.

Match the phase plane plots and the graphs of $x(t)$ and $y(t)$ to the systems below.

Note that in the graphs of $x(t)$ and $y(t)$ vs. t, only the interval $-2 \leq t \leq 2$ is shown. The t intervals for the curves shown in the phase planes overlap this interval, but they may include more or less at either end of the t interval.

(a) $$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = y$$

Phase Plane: ____ **Graphs of $x(t)$ and $y(t)$:** ____

(b) $$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = -2y$$

Phase Plane: ____ **Graphs of $x(t)$ and $y(t)$:** ____

(c) $$\frac{dx}{dt} = -3y \qquad \frac{dy}{dt} = \frac{3}{2}x$$

Phase Plane: ____ **Graphs of $x(t)$ and $y(t)$:** ____

(d) $$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -\frac{1}{2}x - y$$

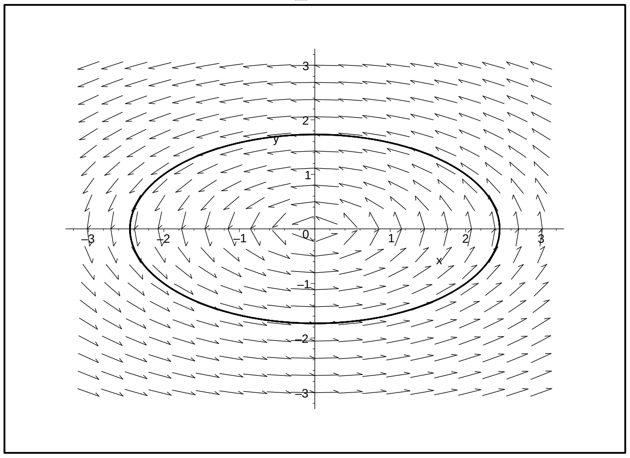
Phase Plane: ____ **Graphs of $x(t)$ and $y(t)$:** ____

(e) $$\frac{dx}{dt} = -x + xy \qquad \frac{dy}{dt} = -y - \frac{1}{2}xy$$

Phase Plane: ____ **Graphs of $x(t)$ and $y(t)$:** ____

Phase Planes for Question 4

1



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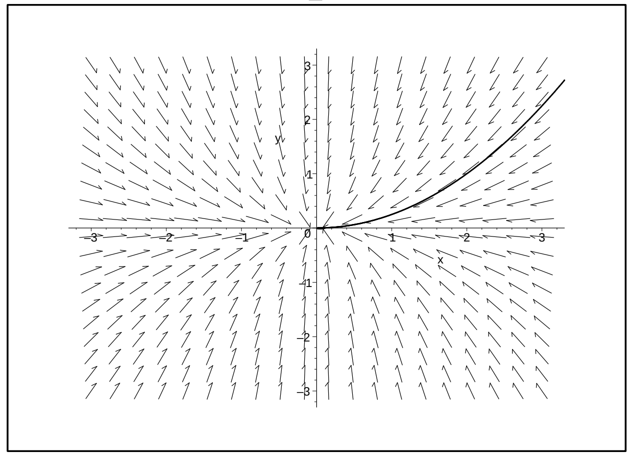
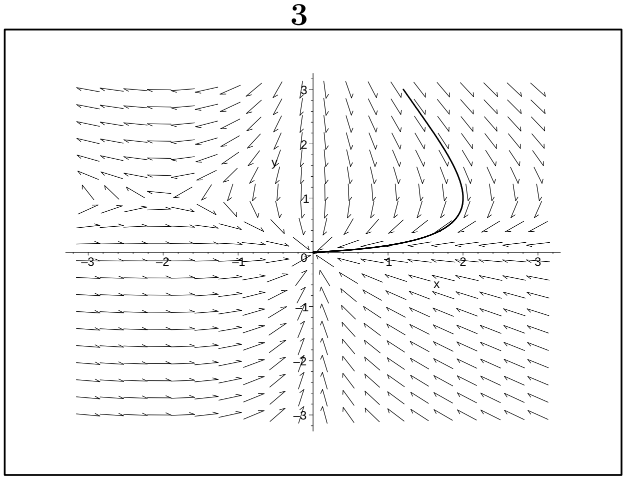


3



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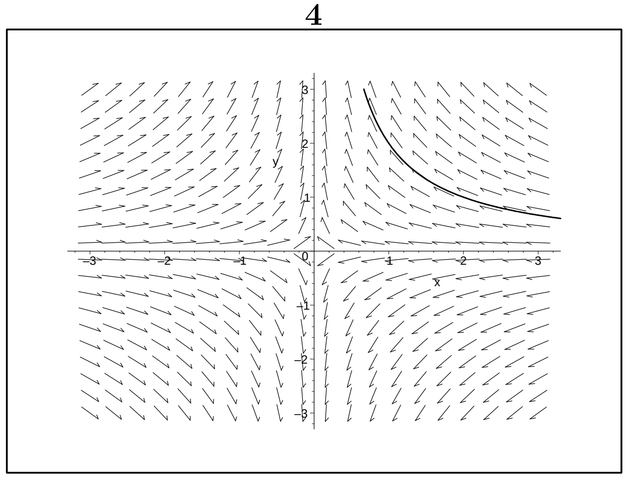
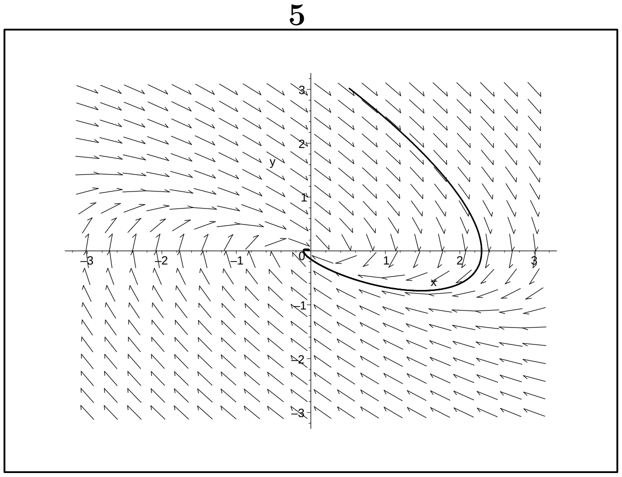
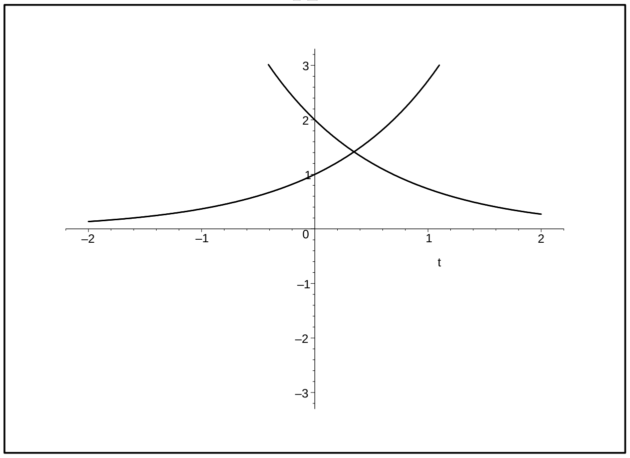


5

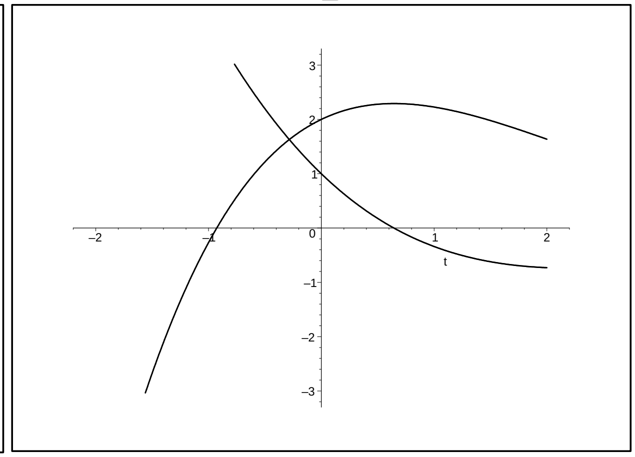


Graphs of $x(t)$ and $y(t)$ versus t for Question 4

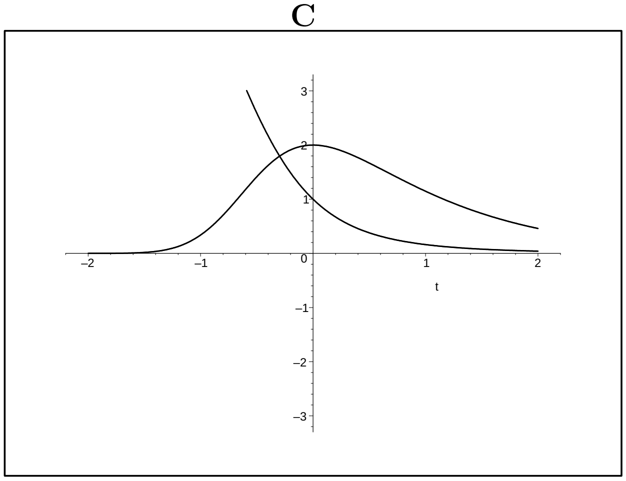
A



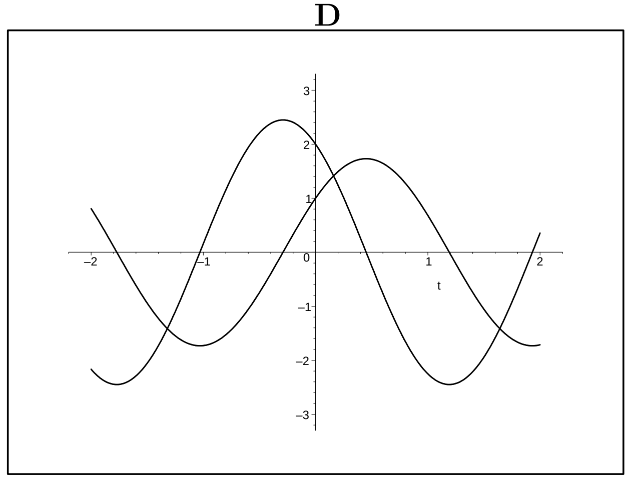
B



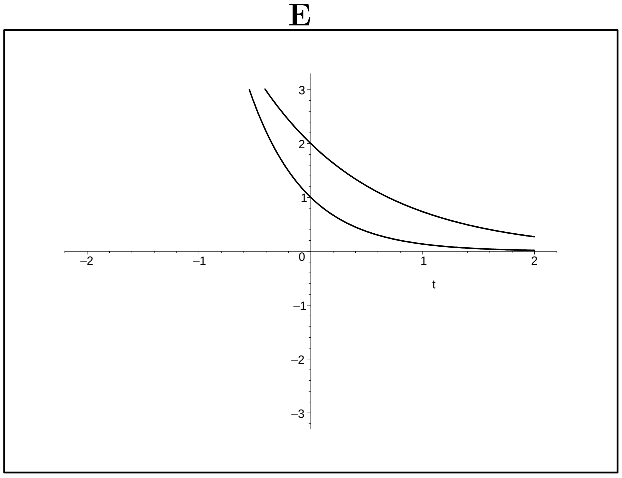
C



D



E



For each of the following systems (5–8),

(a) Find the *real-valued* general solution $\vec{Y}(t)$.

(b) Make a sketch of the phase portrait. Use nullclines (and any other useful lines) to make reasonably accurate sketches of the trajectories in the phase plane.

(c) Classify the type of the equilibrium as a source, sink, saddle, spiral source, spiral sink, center, or "zero eigenvalue".

(d) Solve the initial value problem $\vec{Y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(e) Match the solution to the initial value problem in (d) to one of the ten graphs given in the next two pages. Each graph contains a plot of the components of a vector function $\vec{Y}(t)$ for which $\vec{Y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

5. $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \vec{Y}$

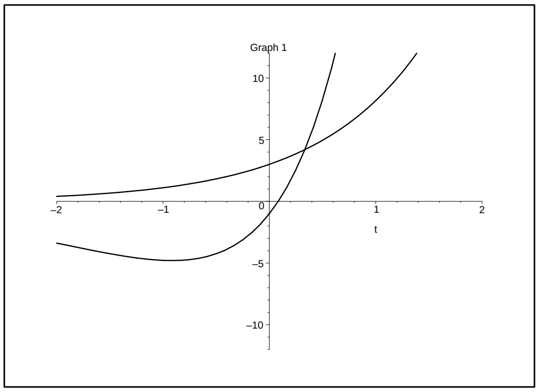
6. $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \vec{Y}$

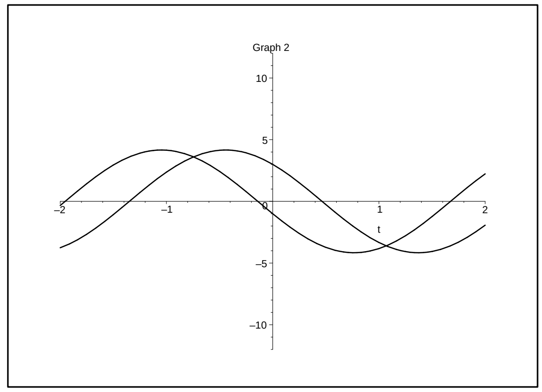
7. $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix} \vec{Y}$

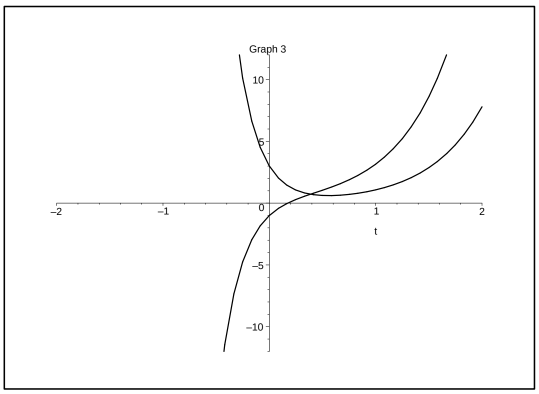
8. $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} 3 & 4 \\ -3 & -4 \end{bmatrix} \vec{Y}$

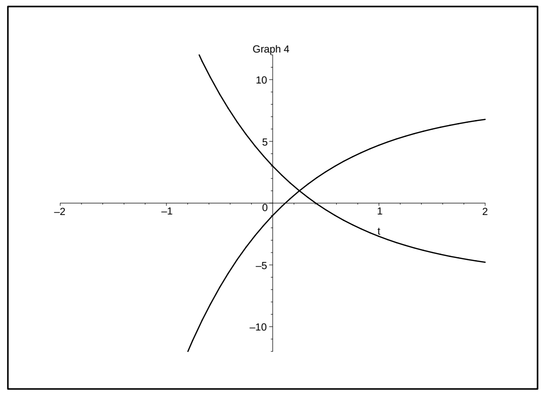
Graphs for Questions 5–8

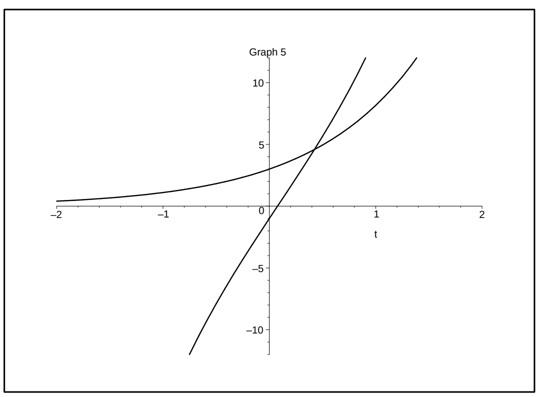
There are more graphs on the next page.

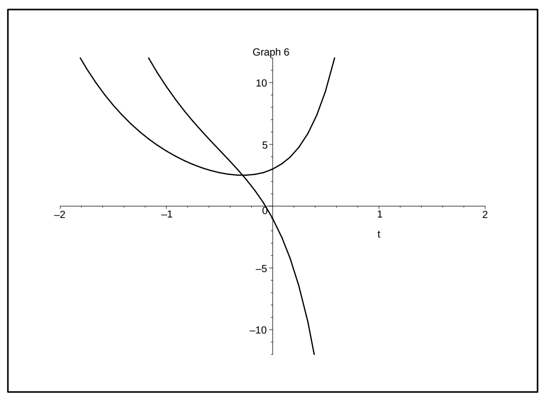


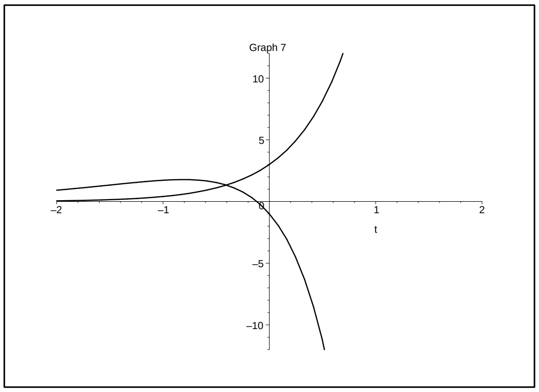


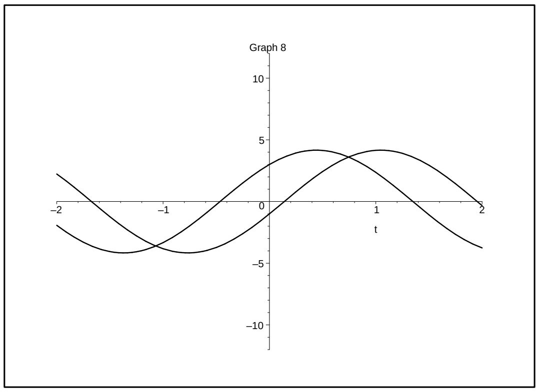


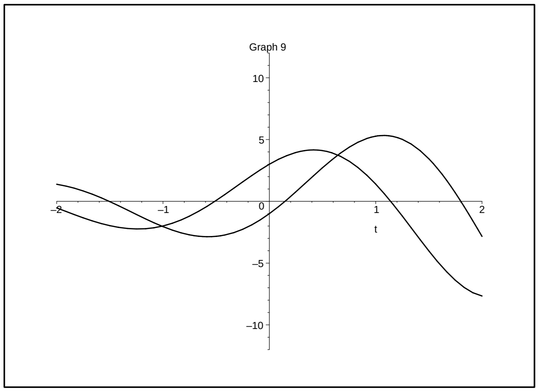


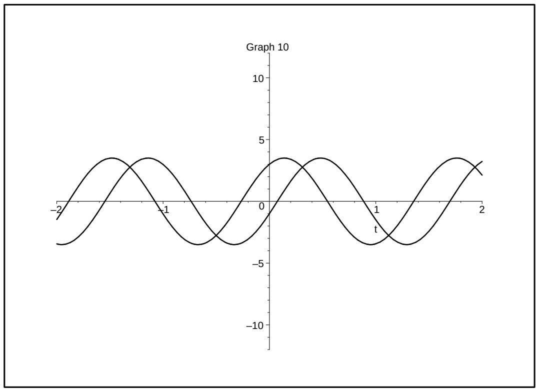





Graph 7
10
5
0
–5
–10
–2
–1
1
2
t


Graph 8
10
5
0
–5
–10
–2
–1
1
2
t


Graph 9
10
5
0
–5
–10
–2
–1
1
2
t


Graph 10
10
5
0
–5
–10
–2
–1
1
2
t

9. Let
$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

   (a) Show that A must have *real* eigenvalues.

   (b) Show that if $a > 0$ and $\det(A) > 0$, then $d > 0$.

   (c) Show that if $a > 0$ and $\det(A) > 0$, then A has *positive* eigenvalues.

10. Consider the following systems. One is a system that models two species, a predator and a prey. Another is a system that models two species that compete for the same resource, so the presence of one species lowers the growth rate of the other species (but neither of the species preys on the other). A third system models two species that have a mutually symbiotic relationship: the presence of one species enhances the growth of the other.

   (a) Identify each system as either "predator/prey", "competing species", or "mutually symbiotic".

   (b) For system (1), find all the equilibrium points.

   (c) Are there steady (i.e. constant in time) population levels in system (1) for which the two species coexist?

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - xy \\ \frac{dy}{dt} &= \frac{2}{3}y(1-2y) - \frac{xy}{2} \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{3}x(1-x/2) + \frac{xy}{6} \\ \frac{dy}{dt} &= y(1-y/3) - \frac{xy}{2} \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{dx}{dt} &= 2x(1-x/2) + \frac{xy}{30} \\ \frac{dy}{dt} &= \frac{1}{2}y(1-y/100) + 3xy \end{aligned} \tag{3}$$
