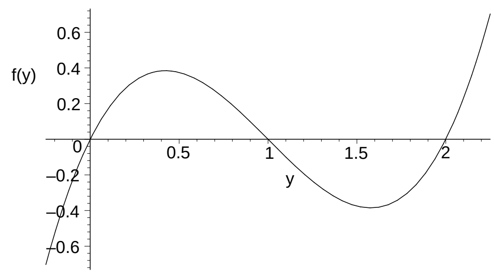
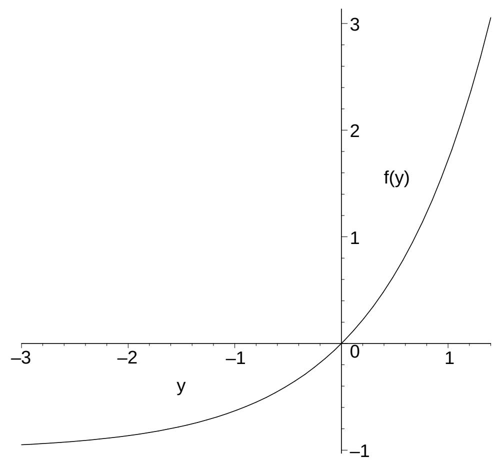
2.5/ 3.



The equilibrium points are $y = 0$, $y = 1$ and $y = 2$. $y = 0$ is unstable; $y = 1$ is asymptotically stable; and $y = 2$ is unstable.

2.5/ 4.

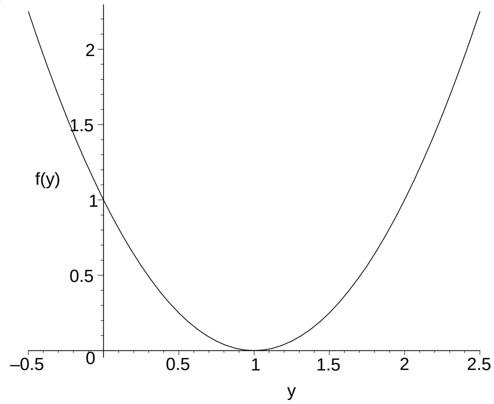


The only equilibrium point is $y = 0$, and it is unstable.

2.5/ 7.

(a) We have $f(y) = k(1-y)^2$. The critical points are the solutions to $f(y) = 0$, and the only solution to $k(1-y)^2 = 0$ (when $k > 0$) is $y = 1$. Thus $y = 1$ is a critical point; the corresponding equilibrium solution is the constant function $\phi(t) = 1$. (Or simply $y(t) = 1$; the book often uses ϕ to refer to specific solutions.)

(b) For any $k > 0$, the graph of $f(y)$ is a parabola the opens upwards, with its minimum at the point $(1, 0)$. For example, here is the plot with $k = 1$:



Clearly $f(y) > 0$ for $y < 0$ and for $y > 0$. Thus, unless $y = 0$, the differential equation $\frac{dy}{dt} = f(y)$ tells us that $y(t)$ must be an increasing function of t.

(c) We already know that $y(t) = 1$ is an equilibrium solution. Now assume $y \neq 1$. By separating, we find

$$\frac{dy}{(1-y)^2} = k\,dt$$

Integrate to obtain

$$\frac{1}{1-y} = kt + C.$$

To satisfy the initial condition $y(0) = y_0$, we must have

$$\frac{1}{1-y_0} = C.$$

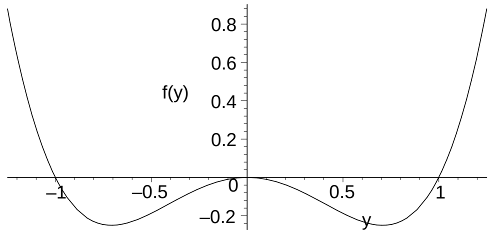
Now solve for y:

$$y = 1 - \frac{1}{kt + C} = 1 - \frac{1}{kt + 1/(1-y0)} = 1 - \frac{1-y_0}{(1-y_0)kt + 1}$$

Consider the term $\frac{1-y_0}{(1-y_0)kt+1}$. If $y_0 < 1$, then the denominator increases monotonically (from the value 1 when $t = 0$), and so the quotient decreases monotonically and approaches 0 asymptotically. Thus $y(t)$ increases monotonically and approaches 1 asymptotically.

If $y_0 > 1$, then the denominator of $\frac{1-y_0}{(1-y_0)kt+1}$ decreases from 1 (when $t = 0$) and becomes 0 when $t = \frac{1}{k(y_0-1)}$. Since the denominator goes to zero, the quotient must "blow up"; and since $1 - y_0 < 0$, it approaches negative infinity. Therefore $y(t)$ is increasing, and has a vertical asymptote at $t = \frac{1}{k(y_0-1)}$.

2.5/ 9.



There are three critical points: $y = -1$, $y = 0$ and $y = 1$.
$y = -1$ is asymptotically stable.
$y = 0$ is semi-stable.
$y = 1$ is unstable.

2.5/ 14. If $f'(y_1) < 0$, then there is an interval containing y_1 such that for all $y < y_1$ in the interval, $f'(y) > 0$, and for all $y > y_1$ in the interval, $f'(y_1) < 0$. This implies that for any solution $y(t)$ that is close to but less than y_1, $y(t)$ is increasing, while for any solution $y(t)$ that is close to but greater than y_1, $y(t)$ is decreasing. Therefore, solutions are are sufficiently close to y_1 must converge to y_1 asymptotically, which means the equilibrium solution $\phi(t) = y_1$ is asymptotically stable.

If $f'(y_1) > 0$, then there is an interval containing y_1 such that for all $y < y_1$ in the interval, $f'(y) < 0$, and for all $y > y_1$ in the interval, $f'(y_1) > 0$. Reasoning as above, this implies that all solutions sufficiently close to y_1 must diverge from y_1. Thus the equilibrium solution $\phi(t) = y_1$ is unstable.

2.5/ 19.

(a) The volume V of a cylinder with constant cross section area A and height h is $V = Ah$. If V and h are functions of time t, then

$$\frac{dV}{dt} = A\frac{dh}{dt}.$$

We are told that water is pumped into the tank at rate k; thus k is a rate of change of the volume V. (We assume that $k > 0$.) The rate at which the water flows out of the hole is $\alpha a\sqrt{2gh}$, which is also a rate of change of the volume. The net rate of change of the volume is the difference of these two quantities. Thus

$$\frac{dV}{dt} = k - \alpha a\sqrt{2gh},$$

or

$$\frac{dh}{dt} = \left(k - \alpha a\sqrt{2gh}\right)/A.$$

(b) Let $f(h) = \left(k - \alpha a\sqrt{2gh}\right)/A$. By solving $f(h) = 0$, we find that the only equilibrium is $h_e = \frac{k^2}{2g\alpha^2 a^2}$. Now $f'(h) = -\frac{\alpha a\sqrt{2g}}{2A\sqrt{h}}$, and $f'(h_e) = -\frac{g\alpha^2 a^2}{kA} < 0$. By the result of problem 14, h_e is asymptotically stable.

2.5/ 20. The differential equation is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey.$$

Let $f(y) = r\left(1 - \frac{y}{K}\right)y - Ey$.

(a) We have $f(y) = -\frac{r}{K}y^2 + (r - E)y$. Solving $f(y) = 0$ gives two solutions: $y_1 = 0$ and $y_2 = K(r - E)/r = K(1 - E/r)$. Actually, this is two distinct solutions if $E \neq r$, but the problem asked for the equilibria under the condition that $E < r$. Also note that if $E < r$, then $y_2 > 0$.

(b) We could compute $f'(y)$ and evaluate at y_1 and y_2, but in this case we can simply point out that the graph of f is a parabola that opens downward, so the slope at $y_1 = 0$ (the left equilibrium) must be postive and the slope at y_2 (the right equilibrium) must be negative. Therefore, by the result of problem 14, y_1 is unstable and y_2 is asymptotically stable.

(c) The sustainable yield is

$$Y = Ey_2 = KE(1 - E/r) = -\frac{K}{r}E^2 + KE.$$

The graph of $Y(E)$ is a parabola opening downwards, with zeros are $E = 0$ and $E = r$.

(d) The maximum of $Y(E)$ occurs when $E = r/2$, and the yield at this value is $Y_m = Y(r/2) = Kr/4$. This is the maximum sustainable yield.

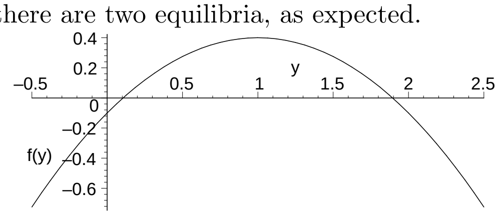
2.5/ 21. The differential equation is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - h.$$

Let $f(y) = r\left(1 - \frac{y}{K}\right)y - h = -\frac{r}{K}y^2 + ry - h$.

(a) By solving $f(y) = 0$ we find $y = -K\left(-r \pm \sqrt{r^2 - 4rh/K}\right)/(2r) = K\left(1 \pm \sqrt{1 - 4h/(rk)}\right)/2$. There are two real distinct solutions if $1 - 4h/rK > 0$, and this holds if $h < rK/4$. Thus, if $h < rK/4$, the equilibrium solutions are $y_1 = K\left(1 - \sqrt{1 - 4h/(rk)}\right)/2$ and $y_2 = K\left(1 + \sqrt{1 - 4h/(rk)}\right)/2$.

A representative graph of $f(y)$ when $h < rK/4$ is shown here. In this example, $r = 1$, $K = 2$ and $h = 0.1$. Since $rK/4 = 0.5$ and $h < 0.5$, there are two equilibria, as expected.



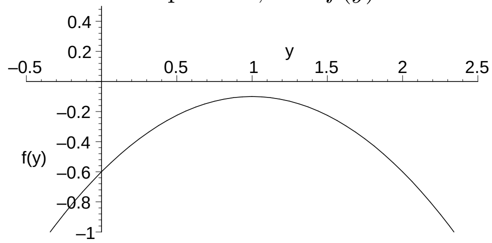
(b) We make a simple argument. The graph of $f(y)$ is a parabola opening downward, so the slope of the graph of f must be positive at y_1 and negative at y_2. Therefore, by problem 14, y_1 is unstable and y_2 is asymptotically stable.

(c) If the initial population y_0 is between y_1 and y_2, then $f(y)$ is always positive, so $y(t)$ will increase and approach y_2 asymptotically. If the $y_0 > y_2$, then $f(y) < 0$, and $y(t)$ will decrease, but it also approaches y_2 asymptotically. Thus for any initial population y_0 larger than y_1, the population will approach y_2 asymptotically.

If $y_0 < y_1$, then $y(t)$ will monotonically decrease, and it will eventually reach zero. (Mathematically, the solution would go through zero and become negative, because zero is *not* an equilibrium solution in this case. However, a negative population is not meaningful. Once the population reaches zero, there are no more fish, so it is pointless to continue from there.)

(d) If $h > rK/4$, then $f(y) = 0$ has no solutions; there are no equilibria. In fact, in this case $f(y) < 0$ for all y. This means that all solutions to the differential equation decrease monotonically, and all solutions will eventually reach zero.

The following plot shows a representative graph of $f(y)$ when $h > rK/4$. In this example, $r = 1$, $K = 2$, and $h = 0.6 > rK/4$. Note that there are no equilibria, and $f(y) < 0$ for all y.



(e) If $h = rK/4$, then $f(y) = -\frac{r}{K}\left(y - \frac{K}{2}\right)^2$. This is that situation discussed in problem 7; the only equilibrium is $y = K/2$ and it is semi-stable.

The following plot shows a representative graph of $f(y)$ when $h = rK/4$. In this example, $r = 1$, $K = 2$, and $h = 0.5 = rK/4$. There is just one equilibrium at $y = 1$, and for all other y, $f(y) < 0$.