Second Order Nonhomogeneous Differential Equations
Three Solved Examples

1. Find the general solution to

$$\frac{d^2y}{dt^2} + 9y = 5\sin 2t.$$

Solution. First find y_h. The homogeneous equation is $\frac{d^2y}{dt^2} + 9y = 0$, so when we guess $y_h = e^{rt}$, we find $r^2 + 9 = 0$. Thus $r = \pm 3i$, and we have

$$y_h = k_1 \cos 3t + k_2 \sin 3t.$$

Now we find the particular solution Y_p. The first guess is

$$Y_p = A\cos 2t + B\sin 2t,$$

and neither term in Y_p solves the homogeneous equation, so this will work. We find

$$Y_p' = -2A\sin 2t + 2B\cos 2t$$
$$Y_p'' = -4A\cos 2t - 4B\sin 2t.$$

Now put these into the original equation:

$$-4A\cos 2t - 4B\sin 2t + 9A\cos 2t + 9B\sin 2t = 5\sin 2t$$
$$(5A)\cos 2t + (5B)\sin 2t = 5\sin 2t,$$

and this implies $A = 0$ and $B = 1$. Thus $Y_p = \sin 2t$. The general solution is

$$y(t) = y_h + Y_p = k_1 \cos 3t + k_2 \sin 3t + \sin 2t.$$

2. Solve the initial value problem is

$$\frac{d^2y}{dt^2} + 4y = 3\cos 2t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution. First find y_h; following the usual procedure, we find $y_h = k_1 \cos 2t + k_2 \sin 2t$.

Now we find the particular solution Y_p. Our first guess is $Y_p = A\cos 2t + B\sin 2t$. This will *not* work, because each term in this guess also solves the homogeneous problem. So we multiply by t to get the next guess:

$$Y_p = At\cos 2t + Bt\sin 2t.$$

Neither term solves the homogeneous problem, so this will work. We find

$$\begin{aligned} Y_p' &= (B - 2At)\sin 2t + (A + 2Bt)\cos 2t \\ Y_p'' &= 4B\cos 2t - 4At\cos 2t - 4A\sin 2t - 4Bt\sin 2t. \end{aligned}$$

Put these into the original equation:

$$4B\cos 2t - 4At\cos 2t - 4A\sin 2t - 4Bt\sin 2t + 4At\cos 2t + 4Bt\sin 2t = 3\cos 2t,$$

so

$$4B\cos 2t - 4A\sin 2t = 3\cos 2t,$$

and we have $A = 0$ and $B = 3/4$. Thus $Y_p = \frac{3}{4}t\sin 2t$.

The general solution is then

$$y(t) = y_h + Y_p = k_1\cos 2t + k_2\sin 2t + \frac{3}{4}t\sin 2t.$$

Now choose k_1 and k_2 to satisfy the initial conditions. $y(0) = 0 \implies k_1 + 0 + 0 = 0 \implies k_1 = 0$. Now

$$y'(t) = 2k_2\cos 2t + (3/2)t\cos 2t + (3/4)\sin 2t,$$

and $y'(0) = 1 \implies 2k_2 + 0 + 0 = 1 \implies k_2 = 1/2$. The solution to the initial value problem is

$$y(t) = \frac{1}{2}\sin(2t) + \frac{3}{4}t\sin 2t.$$

3. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \cos(3t), \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. The characteristic polynomial for the homogeneous equation is $r^2 + 2r + 5$, so we find $r = (-2 \pm \sqrt{4 - 20})/2 = -1 \pm 2i$. Thus

$$y_h = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t).$$

To find Y_p, we could use the formula given in class or in the text, but it is not hard to simply derive it from scratch by using the method of undetermined coefficients. Our first guess is

$$Y_p = A\cos(3t) + B\sin(3t),$$

and this will work because neither term in Y_p also solves the homogeneous problem. We find

$$\begin{aligned} Y_p' &= -3A\sin(3t) + 3B\cos(3t) \\ Y_p'' &= -9A\cos(3t) - 9B\sin(3t). \end{aligned}$$

Substituting Y_p into the differential equation gives

$$-9A\cos(3t) - 9B\sin(3t) + 2\left(-3A\sin(3t) + 3B\cos(3t)\right) + 5(A\cos(3t) + B\sin(3t)) = \cos(3t),$$

or

$$(-4A + 6B)\cos(3t) + (-4B + 6A)\sin(3t) = \cos(3t).$$

This implies $-4A + 6B = 1$ and $-4B + 6A = 0$; solving these for A and B gives $A = -1/13$ and $B = 3/26$. The general solution is therefore

$$y(t) = y_h + Y_p = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) - \frac{1}{13}\cos(3t) + \frac{3}{26}\sin(3t).$$

Now we must determine c_1 and c_2 to satisfy the initial conditions. First, we'll need $y'(t)$:

$$y'(t) = c_1(-2e^{-t}\sin(2t) - e^{-t}\cos(2t)) + c_2(2e^{-t}\cos(2t) - e^{-t}\sin(2t)) + \frac{3}{13}\sin(3t) + \frac{9}{26}\cos(3t).$$

The given initial conditions $y(0) = 1$ and $y'(0) = 0$ imply

$$\begin{aligned} c_1 - \frac{1}{13} &= 1 \\ -c_1 + 2c_2 + \frac{9}{26} &= 0 \end{aligned}$$

and we find $c_1 = 14/13$ and $c_2 = 19/52$. Thus the solution to the initial value problem is

$$y(t) = \frac{14}{13}e^{-t}\cos(2t) + \frac{19}{52}e^{-t}\sin(2t) - \frac{1}{13}\cos(3t) + \frac{3}{26}\sin(3t).$$