Summary of the Method of Undetermined Coefficients

The Method of Undetermined Coefficients is a method for finding a particular solution to the second order nonhomogeneous differential equation

$$ay'' + by' + cy = g(t)$$

when $g(t)$ has a special form, involving only polynomials, exponentials, sines and cosines.

In the following table, $P_n(t)$ is a polynomial of degree n: $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

$g(t)$	$Y_p(t)$ (first guess)
ke^{rt}	Ae^{rt}
$k\cos(\omega t)$ **or** $k\sin(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$P_n(t)$	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
$P_n(t)e^{rt}$	$\big(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\big)e^{rt}$
$P_n(t)e^{rt}\cos(\omega t)$ **or** $P_n(t)e^{rt}\sin(\omega t)$	$\big(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0\big)e^{rt}\cos(\omega t) +$ $\big(B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0\big)e^{rt}\sin(\omega t)$

If any term in the first guess is also a solution to the corresponding homogeneous equation, multiply the *whole guess* by t. If any term in this second guess is still a solution to the homogeneous equation, multiply by t again (i.e. multiply the first guess by t^2).

To find the coefficients, substitute Y_p into the differential equation, and collect the coefficients of the different functions of t.

Example. Consider $y'' + 3y' + 2y = t^2$. We find $y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$. Since $g(t) = t^2$, a second degree polynomial, we use the third line in the above table, and we guess $Y_p(t) = A_2 t^2 + A_1 t + A_0$. None of the three terms in this guess also solves the homogeneous equations, so this guess will work.

Example. Consider $y'' + 6y' + 10y = te^{-3t}\cos(t)$. We find $y_h(t) = c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t)$. Now $g(t) = te^{-3t}\cos(t)$, so we use the fifth line in the table above ($n = 1$, $a_1 = 1$, $a_0 = 0$, $r = -3$, $\omega = 1$) to make the first guess

$$\begin{aligned} Y_p(t) &= (A_1 t + A_0)e^{-3t}\cos(t) + (B_1 t + B_0)e^{-3t}\sin(t) \\ &= A_1 te^{-3t}\cos(t) + A_0 e^{-3t}\cos(t) + B_1 te^{-3t}\sin(t) + B_0 e^{-3t}\sin(t). \end{aligned}$$

However, the terms $A_0 e^{-3t}\cos(t)$ and $B_0 e^{-3t}\sin(t)$ both solve the homogeneous equation, so we must multiply the first guess by t. Our guess is now

$$\begin{aligned} Y_p(t) &= t\left\{(A_1 t + A_0)e^{-3t}\cos(t) + (B_1 t + B_0)e^{-3t}\sin(t)\right\} \\ &= A_1 t^2 e^{-3t}\cos(t) + A_0 te^{-3t}\cos(t) + B_1 t^2 e^{-3t}\sin(t) + B_0 te^{-3t}\sin(t). \end{aligned}$$

None of the terms in this guess solves the homogeneous equation, so this guess will work.