Math 308 - Differential Equations **Fall 2003**

Homework Assignment 1

Due *Wednesday, September 24.*

Note: Some of these problems will require numerical calculations. In calculations requiring several steps, keep at least four significant digits in the intermediate steps. Use a computer program (such as Maple or MathCAD) to create and print any plots. Be sure that in each plot the axes are clearly labeled, each curve is labeled (if there is more than one curve), and there is a meaningful title.

1. **A Preliminary Problem.** This problem presents the solution to a linear *difference* equation, rather than a differential equation. The result will be useful in Problem 2. Consider the difference equation

$$x_{n+1} = px_n + q, \qquad n \geq 1, \quad \text{and } x_0 \text{ given} \tag{1}$$

where x_n, $n = 1, 2, 3, \ldots$ are unknown, and p and q are constants. For our purposes it will be sufficient to assume that $|p| < 1$. An equation like this shows up in Problem 2 below.

   (a) Verify that $x_n = p^n \left(x_0 - \frac{q}{1-p} \right) + \frac{q}{1-p}$ solves the difference equation (1) for any value of x_0.

   (b) Find $\lim\limits_{n \to \infty} x_n$.

2. **Drugs.** A common model for drug absorption in the body is that, after receiving a dose (by an injection, say), the drug will be cleared from the bloodstream at a rate that is proportional to the amount present. In other words, if $a(t)$ is the amount of the drug in the bloodstream at time t (in hours), then

$$\frac{da}{dt} = -ka,$$

where $k > 0$ is the proportionality constant.

   (a) Suppose that at $t = 0$, a person (whose bloodstream is free of the drug for $t < 0$) is given 80.00 units of the drug, and 24 hours later the amount of the drug remaining in the bloodstream is found to be 12.50 units. Use this data to determine k.

   (b) After the first 24 hours, the patient is again given a dose of 80 units. What will be the amount of the drug in the bloodstream at the end of the next 24 hours?

   (c) Suppose the patient continues to receive 80 units every 24 hours. Let a_n be the amount of the drug in the bloodstream at the end of the nth 24 hour period. Determine the difference equation satisfied by a_n, and then use the formula given in Problem 1 to find the solution to this difference equation.

   (d) What is the limiting value of the amount of drug in the bloodstream at the end of the nth 24 hour period as n increases?

3. **The Wind on Wynn.** An anemometer is a device that measures wind speed. A common type of anemometer is the "spinning cup" anemometer, like the one on top of Wynn Hall. In this problem we consider a simple model of an anemometer. In the simplest sort of anemometer, the wind speed displayed by the device is simply a multiple of the rate at which it is spinning. The question we will address here is whether that is a reasonable design. First we need to set up our mathematical model.

   Newton's second law of motion says $F = ma$, where F is the net force applied to an object, a is the acceleration of the object, and m is the (constant) mass of the object. (See Example 1 in the text, page 2.) There is an analogous formula for *rotating motion.* This is

$$T = I\frac{d\omega}{dt},$$

where T is the *torque*, I is the *moment of inertia* of the anemometer, ω is the *angular velocity*[1] (i.e the spin rate), and $\frac{d\omega}{dt}$ is the angular acceleration. The units of ω might be, for example, radians per second.

In our model of an anemometer, we'll include two torques. These result from friction and from the force of the wind. The simplest model of friction is that the frictional torque is proportional to the angular velocity, and since friction "opposes" the angular velocity, the frictional torque has the form

$$T_f = -k\omega,$$

where $k > 0$ is a constant. The simplest model of how the wind acts on the anemometer is that the torque due to the wind, T_v, is proportional to the wind speed v: $T_v = rv$ for some constant $r > 0$. Then the total torque is $T = T_f + T_v = -k\omega + rv$. Newton's law gives

$$\frac{d\omega}{dt} = \frac{1}{I}\left(-k\omega + rv\right).$$

The constant I would have to be computed (based on the shape and mass of the anemometer), and the constants k and r would have to be determined experimentally. Let us suppose that someone has already determined that, when $v(t)$ is expressed in meters per second, and ω is in radians per second, we have $k/I = 2$ and $r/I = 1$. Then the differential equation becomes

$$\frac{d\omega}{dt} = -2\omega + v(t). \tag{2}$$

(a) Suppose (as mentioned earlier), the anemometer simply displays a multiple of the angular velocity as an approximation of the wind speed. That is, while the true velocity might be $v(t)$, the display will be $W(t) = c\omega(t)$, where c is a conversion constant. The constant is chosen so that in a perfectly steady wind (perhaps tested in a wind tunnel), $W(t)$ gives the correct wind speed.

By assuming that $v(t) = v_1$, a constant, determine the value of c so that $W(t) = c\omega(t)$ is the correct steady state wind speed. Use this constant c in the rest of the problem.

(b) Suppose that the anemometer is at rest for $t < 0$, so $\omega(0) = 0$. At $t = 0$, the wind speed suddenly jumps to $v(t) = 3$ meters per second.

i. Find the resulting angular velocity $\omega(t)$, and find the error $v(t) - W(t)$.

ii. What happens to the error as t increases?

iii. Calculate how long it will take for $W(t)$ to reach 90% of the steady state value.

iv. On the same set of axes, plot $W(t)$ and $v(t)$ for $0 \le t \le 5$.

(c) Let's see what happens in a steadily changing wind speed. Suppose that $\omega(0) = 0$, and $v(t) = mt$. (The parameter m determines how fast the wind speed is changing.)

i. Find $\omega(t)$, and find the error $v(t) - W(t)$. Your answers will depend on m.

ii. Discuss how the error depends on m. Also discuss what happens as t increases. Does the error decrease? Does it go to zero?

iii. Suppose $m = 0.5$. In one graph, plot $v(t)$, $W(t)$, and the error $v(t) - W(t)$ for $0 \le t \le 5$.

(d) Suppose the wind is blowing in periodic gusts. Let $v(t) = 1 - \cos(kt)$, where k is a constant that determines the frequency of the gusts.

i. Find $\omega(t)$, and find the error $v(t) - W(t)$. Your answers will depend on k.

ii. What happens to the error as t increases? Does the error go to zero? How does the error depend on k?

iii. Suppose $k = 1/3$. In one graph, plot $v(t)$, $W(t)$, and $|v(t) - W(t)|$ for $0 \le t \le 30$.

[1] "ω" is the lower case Greek letter "omega", not a fancy "w".

iv. Suppose $k = 2$. In one graph, plot $v(t)$, $W(t)$, and $|v(t) - W(t)|$ for $0 \le t \le 30$.

(e) Does using a multiple of the angular velocity as an approximation of the wind speed seem reasonable? Under what wind conditions is the approximation appropriate, and what conditions make it a bad approximation?

4. **An Economics Model.** In this problem we look at a model from macroeconomics. Let K be the capital[2], L the labor, and Q the production output. We are interested in a *dynamic* problem, so $K(t)$, $L(t)$ and $Q(t)$ are all functions of time, but we will suppress the t argument. In elementary economics, one learns that a common assumption is that Q can be expressed as function of K and L:

$$Q = f(K, L). \tag{3}$$

We make the reasonable assumptions that $f_K > 0$ and $f_L > 0$. (The subscript denotes a partial derivative: $f_K = \partial f / \partial K$.) These assumptions mean that Q increases if either K or L increases. That is, with more capital or more labor, we can produce more. We also assume that $f_{KK} < 0$ and $f_{LL} < 0$. These assumptions say that f has *diminishing returns* to the inputs K and L. In other words, the larger K is, the less is the effect of increasing K, and the same holds for L. Finally, we also assume that f has, using economics terminology, *constant returns to scale.* Mathematically, this means that multiplying K and L by the same amount results in Q being multiplied by the same amount. That is, for any constant b,

$$f(bK, bL) = bf(K, L).$$

For example, the Cobb-Douglas function $f(K, L) = K^{1/3} L^{2/3}$ satisfies the above assumptions. (You should check this, but you don't have to show it in your work.)

We make two more assumptions. We assume that a constant proportion of Q is invested in capital. This means that the *rate of change* of K is proportional to Q:

$$\frac{dK}{dt} = sQ, \tag{4}$$

where s is the proportionality constant. We also assume that the labor force is growing according to the equation

$$\frac{dL}{dt} = \lambda L, \tag{5}$$

where $\lambda > 0$ is the growth rate. (As you know, this is a simple first order equation for L which we can solve to find $L = L_0 e^{\lambda t}$.)

Now we will combine equations (3), (4), and (5) to obtain a first order differential equation. First we express the relation $Q = f(K, L)$ in a different form. Because f has constant returns to scale, we can write

$$Q = f(K, L) = f\left(L\frac{K}{L}, L\right) = Lf\left(\frac{K}{L}, 1\right) = Lg(k), \tag{6}$$

where we have introduced a new variable $k = K/L$ (so k is the *ratio* of capital to labor), and we have defined a new function $g(k) = f(k, 1)$. By differentiating the relation $K = kL$ and then using equation (5) we obtain

$$\begin{aligned} \frac{dK}{dt} &= k\frac{dL}{dt} + \frac{dk}{dt}L \\ &= k(\lambda L) + \frac{dk}{dt}L \\ &= \left(\lambda k + \frac{dk}{dt}\right)L \end{aligned}$$

[2] *Capital* includes things that are owned to be used in production, such as buildings and manufacturing equipment.

Then equation (4) gives us

$$
\begin{aligned}
Q &= \frac{1}{s}\frac{dK}{dt} \\
&= \frac{1}{s}\left(k\lambda + \frac{dk}{dt}\right)L
\end{aligned}
$$

Finally, we put this expression for Q into (6):

$$\frac{1}{s}\left(k\lambda + \frac{dk}{dt}\right)L = Lg(k).$$

Cancel L, and rearrange a bit to obtain the first order differential equation for k:

$$\frac{dk}{dt} = -\lambda k + sg(k).$$

This equation is the *Solow growth model.*

(a) As a concrete example, let's take the production function to be a Cobb-Douglas function $f(K,L) = K^{1/3}L^{2/3}$. Under this assumption, what is $g(k)$, and what is the differential equation for k?

(b) Keeping in mind that λ and s are *positive* constants, do a qualitative analysis of the differential equation in part (a). (Since K and L are positive, you may assume that $k \geq 0$.) Include in your analysis any equilibrium solutions, and discuss the behavior of k as time increases. What if $k(0)$ is near zero, or what if $k(0)$ is large? In addition to describing how k behaves, be sure to explain what this means in terms of the capital K, the labor L, and the production Q.

(c) What roles do the constants s and λ play in your analysis? Does changing s or λ change the number of equilibria, or the long term behavior of the solutions? How do changes in s or λ affect the solutions?

(d) Do you expect your qualitative analysis would be similar for other production functions (as long as they satisfy the assumptions on f given earlier)? Explain.

5. **Balloon Science?** Air is pumped into a spherical balloon at a constant rate of 12 cubic centimeters per second.

(a) Find the differential equation for the volume v of the balloon.

(b) Find the differential equation for the radius r of the balloon.

(c) Solve the differential equations from (a) and (b). You can solve them individually, or you can use the solution of one to find the solution of the other.

(The volume of a sphere with radius r is $4\pi r^3/3$.)