Math 308 - Differential Equations **Fall 2003**

Homework 1 Solutions

1. The difference equation is

$$x_{n+1} = px_n + q, \qquad n \geq 1, \quad \text{and } x_0 \text{ given} \tag{1}$$

where $|p| < 1$.

(a) We must show that $x_n = p^n \left(x_0 - \frac{q}{1-p}\right) + \frac{q}{1-p}$ solves the difference equation (1) for any value of x_0. We have

$$\begin{aligned} px_n + q &= p\left\{p^n \left(x_0 - \frac{q}{1-p}\right) + \frac{q}{1-p}\right\} + q \\ &= p^{n+1}\left(x_0 - \frac{q}{1-p}\right) + \frac{pq}{1-p} + q \\ &= p^{n+1}\left(x_0 - \frac{q}{1-p}\right) + \frac{pq + (1-p)q}{1-p} \\ &= p^{n+1}\left(x_0 - \frac{q}{1-p}\right) + \frac{q}{1-p} \\ &= x_{n+1}, \end{aligned}$$

so the formula does give the solution.

(b) Because $|p| < 1$, $\lim_{n\to\infty} |p|^n = 0$. Thus $\lim_{n\to\infty} x_n = \dfrac{q}{1-p}$.

2. The amount of the drug in the bloodstream, $a(t)$, is determined by

$$\frac{da}{dt} = -ka,$$

where $k > 0$ is the proportionality constant.

(a) The general solution to the differential equation is $a(t) = Ce^{-kt}$, and to satisfy $a(0) = 80$, we must have $C = 80$, so $a(t) = 80e^{-kt}$. We are given $a(24) = 12.50$, so $12.50 = 80e^{-24k}$. Solving for k gives $k = 0.07735$.

(b) At the moment after the second dose is given, the amount of the drug in the bloodstream is 92.50 (12.50 remains from the first dose, and 80.00 is added by the second dose). After the second dose, the drug continues to obey the same differential equation, and therefore the amount of the drug at the end of the second 24 hour period is $92.50e^{-24k} = 14.45$.

(c) We have

$$\begin{aligned} a_2 &= (a_1 + 80)e^{-24k} = 14.45, \\ a_3 &= (a_2 + 80)e^{-24k} = 14.76, \\ a_4 &= (a_3 + 80)e^{-24k} = 14.81, \\ &\text{etc.} \end{aligned}$$

and in general, we have

$$a_{n+1} = (a_n + 80)e^{-24k} = e^{-24k}a_n + 80e^{-24k}.$$

This is a difference equation of the type discussed in problem 1, with $p = e^{-24k} = 0.1563$, and $q = 12.50$. Given the meaning of a_n, we see that $a_0 = 0$. Therefore the solution is

$$\begin{aligned} a_n &= p^n \left(\frac{-q}{1-p} \right) + \frac{q}{1-p} \\ &= e^{-24kn} \frac{-12.50}{1-e^{-24k}} + \frac{12.50}{1-e^{-24k}} \\ &= \frac{12.50}{1-e^{-24k}} \left(1 - e^{-24kn}\right) \\ &= (14.82)\left(1 - (0.1563)^n\right) \end{aligned}$$

(d) As $n \to \infty$, $a_n \to \dfrac{q}{1-p} = \dfrac{12.50}{1-e^{-24k}} = 14.82$.

3. The differential equation for $\omega(t)$ is

$$\frac{d\omega}{dt} = -2\omega + v(t). \tag{2}$$

(a) When $v(t) = v_1$, the differential equation is

$$\frac{d\omega}{dt} = -2\omega + v_1.$$

This equation has a single asymptotically stable equilibrium solution $\omega(t) = v_1/2$. Thus, after a long time, the display of the anemometer will be $W(t) = c\omega(t) = cv_1/2$. The actual wind speed is v_1, so we must choose $c = 2$ in order to have $W(t) = v(t)$.

(b) Under these conditions, $\omega(t)$ is the solution to the initial value problem

$$\frac{d\omega}{dt} = -2\omega + 3 \quad (t > 0), \quad \text{and} \quad \omega(0) = 0.$$

i. The solution to this initial value problem is

$$\omega(t) = -\frac{3}{2}e^{-2t} + \frac{3}{2},$$

and the error is

$$v(t) - W(t) = 3 - 2\omega(t) = 3e^{-2t}.$$

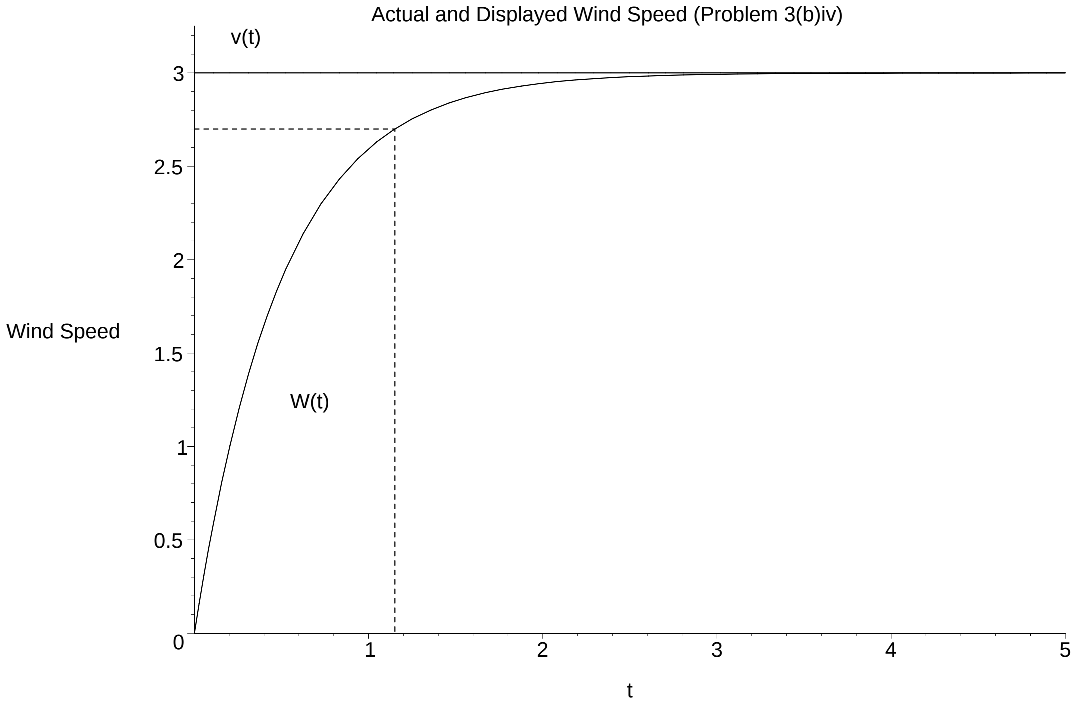
ii. We see that as t increases, the error decays to zero exponentially. Initially, the display of the wind speed will be inaccurate; it takes some time for the anemometer to "spin up" towards the steady state.

iii. The steady state value of $W(t)$ is 3, and 90% of this 2.7. Let T be the time at which $W(t)$ reaches 2.7; that is, $W(T) = 2\omega(T) = -3e^{-2T} + 3 = 2.7$. Solving for T gives

$$T = \ln(10)/2 = 1.151.$$

Thus it will take 1.151 seconds for the display to be within 90% of the correct value.

iv. The following plot shows $W(t)$ and $v(t)$. The point where $W(t) = 2.7$ is also indicated.



(c) In this case, the initial value problem for $\omega(t)$ is

$$\frac{d\omega}{dt} = -2\omega + mt \quad (t > 0), \quad \text{and} \qquad \omega(0) = 0.$$

i. The differential equation is linear, with $p(t) = 2$ and $g(t) = mt$, so we use the integrating factor formula to solve it. We have

$$\mu(t) = e^{\int p(t)\,dt} = e^{2t},$$

and

$$\begin{aligned}\omega(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right)\\ &= e^{-2t}\left(\int e^{2t}mt\,dt + C\right)\\ &= e^{-2t}\left(m\left(\frac{2t-1}{4}\right)e^{2t} + C\right)\\ &= m\left(\frac{2t-1}{4}\right) + Ce^{-2t}\end{aligned}$$

To satisfy the initial condition $\omega(0) = 0$, we have $\omega(0) = -\frac{m}{4} + C = 0$, so $C = \frac{m}{4}$. The solution to the initial value problem is

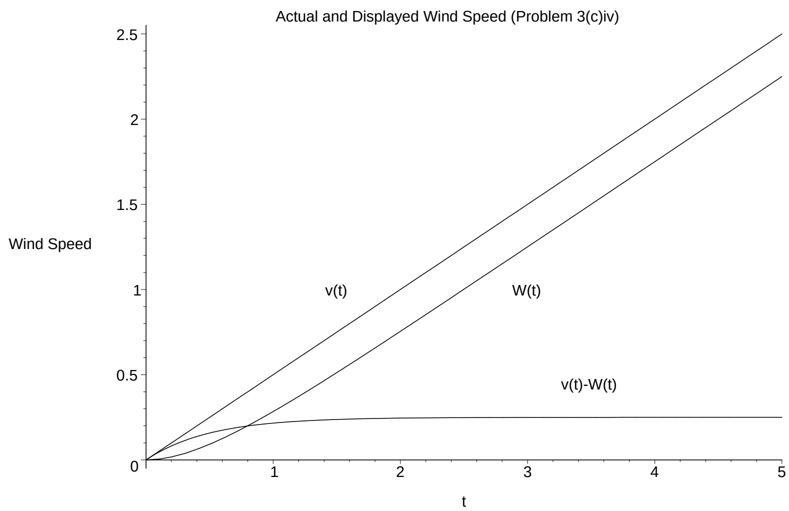
$$\omega(t) = m\left(\frac{2t-1}{4}\right) + \frac{m}{4}e^{-2t}.$$

and the error is

$$v(t) - W(t) = \frac{m}{2}\left(1 - e^{-2t}\right).$$

ii. The error is proportional to m, so the faster the wind speed increases, the greater the error will be. As t increases, the exponential term in the error will approach zero, but the term $m/2$ remains. Thus, in a steadily increasing wind, the error does not go to zero, and instead it approaches the steady-state value of $m/2$. Roughly speaking, the anemometer can not keep up with the changing wind speed.

iii.



(d) In this case, the differential equation for $\omega(t)$ is

$$\frac{d\omega}{dt} = -2\omega + 1 - \cos(kt),$$

i. We now have a linear equation with $p(t) = 2$ and $g(t) = 1 - \cos(kt)$. The integrating factor is still $\mu(t) = e^{2t}$, and

$$\begin{aligned}
\omega(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right) \\
&= e^{-2t}\left(\int e^{2t}(1 - \cos(kt)\,dt + C\right) \\
&= e^{-2t}\left(\frac{e^{2t}}{2} - \frac{2\cos(kt) + k\sin(kt)}{4+k^2}\,e^{2t} + C\right) \\
&= \frac{1}{2} - \frac{2\cos(kt) + k\sin(kt)}{4+k^2} + Ce^{-2t}.
\end{aligned}$$

The error is

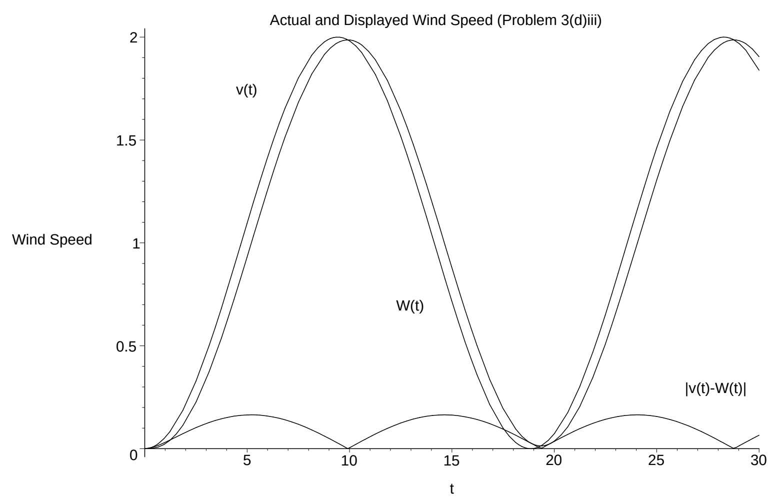
$$\begin{aligned}
v(t) - W(t) &= 1 - \cos(kt) - 2\omega(t) \\
&= -\cos(kt) + 2\frac{2\cos(kt) + k\sin(kt)}{4+k^2} - 2Ce^{-2t} \\
&= \frac{2k\sin(kt) - k^2\cos(kt)}{4+k^2} - 2Ce^{-2t}
\end{aligned}$$

ii. As t increases, the exponential term in the error will go to zero, but the term involving $\sin(kt)$ and $\cos(kt)$ will not. The error will continue to oscillate, with an amplitude that depends on k. If k is "large" (e.g. much larger than 2), then the dominant term in the error is $\frac{-k^2\cos(kt)}{4+k^2} \approx -\cos(kt)$, so the long term error is an oscillation with an amplitude of approximately 1. If k is "small" (much less than 2), then the dominant term in the error is $\frac{2k\sin(kt)}{4+k^2} \approx \frac{k\sin(kt)}{2}$, so the long term error is an oscillation with an amplitude of approximately $k/2$.

iii. When $k = 1/3$ we have

$$\begin{aligned} v(t) &= 1 - \cos(t/3) \\ \omega(t) &= \frac{1}{2} - \frac{18}{37}\cos(t/3) - \frac{3}{37}\sin(t/3) + Ce^{-2t} \\ |v(t) - W(t)| &= \left| -\frac{1}{37}\cos(t/3) + \frac{6}{37}\sin(t/3) - 2Ce^{-2t} \right| \end{aligned}$$

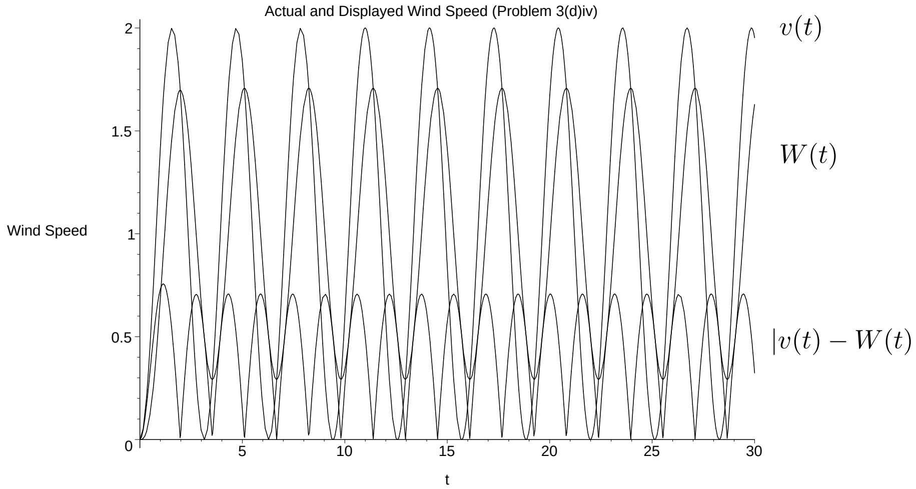
It wasn't stated in the problem, but in this part and the next, I will take the initial condition to be $\omega(0) = 0$. To satisfy this initial condition, choose $C = -1/74$. The following plot shows the graphs. Note that the amplitude of the oscillation of the error is 0.16.



iv. When $k = 2$ we have

$$\begin{aligned} v(t) &= 1 - \cos(2t) \\ \omega(t) &= \frac{1}{2} - \frac{1}{4}\cos(2t) - \frac{1}{4}\sin(2t) + Ce^{-2t} \\ |v(t) - W(t)| &= \left| -\frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t) - 2Ce^{-2*t} \right| \end{aligned}$$

I'll choose $C = -1/4$ so that $\omega(0) = 0$. The following plot shows the graphs. Note that the amplitude of the oscillation of the error is 0.70, over four times the amplitude of the case where $k = 1/3$.



(e) Using a multiple of the angular velocity as an approximation of the wind speed is reasonable if the wind speed is constant or changing slowly. When the wind speed is constant, the error will go to zero, and when the wind speed changes slowly, the error is (roughly) proportional to the rate of change of the wind speed.

When the wind speed is changing rapidly (for example, when m is large in (c) or when k is large in (d)), the angular velocity of the anemometer can not keep up with the wind speed, and there will be large errors.

4. (a) When $f(K, L) = K^{1/3}L^{2/3}$, we have

$$g(k) = f(k, 1) = k^{1/3}.$$

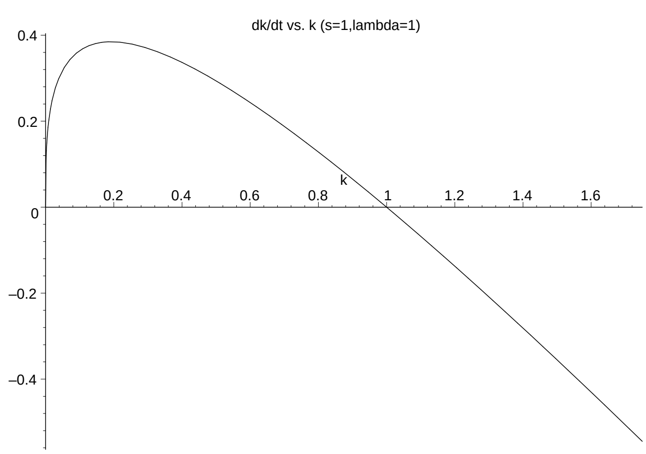
The differential equation for k is

$$\frac{dk}{dt} = -\lambda k + sk^{1/3}.$$

(b) First we find the equilibrium solutions. By solving

$$-\lambda k + sk^{1/3} = 0,$$

we find $k = 0$ or $k = (s/\lambda)^{3/2}$. Note that $\frac{d(-\lambda k + sk^{1/3})}{dk} = -\lambda + \frac{s}{3}k^{-2/3}$, which becomes infinite when $k = 0$, and is zero when $k = \left(\frac{s}{3\lambda}\right)^{3/2}$ (which is between the two equilibrium points). The following plot shows the graph of $\frac{dk}{dt}$ versus k when $s = 1$ and $\lambda = 1$.

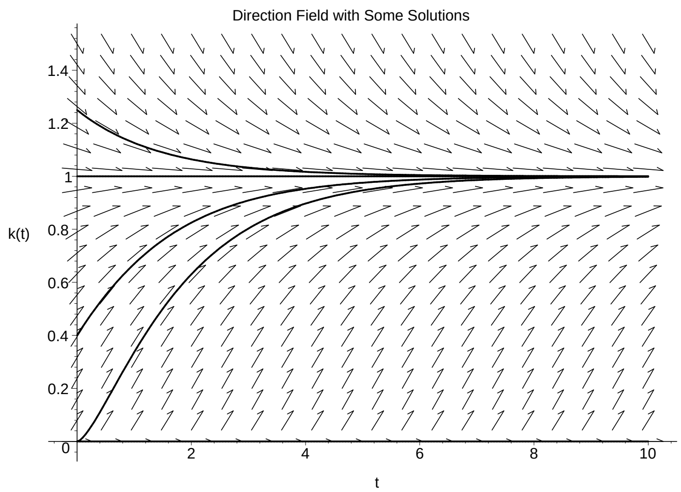


Changing λ or s will change the scale (and the numerical value of the non-zero equilibrium), but the graph of dk/dt versus k will always have the shape as the graph shown above.

We see that the equilibrium $k = 0$ is *unstable*, and if k is small, k will increase rapidly. The graph of $k(t)$ will have an inflection point when k reaches $\left(\frac{s}{3\lambda}\right)^{3/2}$ (where dk/dt has its maximum). k will then converge asymptotically to the non-zero equilibrium.

The equilibrium $k = (s/\lambda)^{3/2}$ is *asymptotically stable*. In fact, *all* solutions with $k(0) > 0$ will converge asymptotically to this equilibrium.

The following shows a direction field and some solutions when $s = 1$ and $\lambda = 1$.



Since $k(t) = K(t)/L(t)$, and $L(t) = L_0 e^{\lambda t}$, if $k(t)$ converges to an asymptotically stable equilibrium k_1, then $K(t)$ must behave asymptotically like $k_1 L(t)$. This means that, in the long term, $K(t)$ must grow exponentially, with the same exponent as $L(t)$. This model predicts that in the long term, production will grow exponentially along with the labor. If, for example, production is too low, it will rapidly increase until it becomes proportional to the labor, and then it will settle into a long term behavior in which production remains proportional to the labor.

(c) As shown above, there will always be two equilibria when $s > 0$ and $\lambda > 0$, and they will always have the stability properties described above. Changing s or λ changes the value of the non-zero equilibrium point, and also changes the rates at which solutions will grow or decay, but the qualitative picture does not change when s or λ are changed.

(d) A more general version of the Cobb-Douglas function that has constant returns to scale is $f(L, K) = L^{\alpha} K^{1-\alpha}$, where $0 < \alpha < 1$. This production function would result in the same qualitative analysis as the one given above. It results in $g(k) = f(k, 1) = k^{\alpha}$, and the differential equation for k is

$$\frac{dk}{dt} = -\lambda k + s k^{\alpha}.$$

This equation has an unstable equilibrium at $k = 0$, and an asymptotically stable equilibrium at $k = \left(\frac{s}{\lambda}\right)^{\frac{1}{1-\alpha}}$.

5. (a) Let $v(t)$ be the volume of the balloon (in cm^3) at time t (in seconds). Then the differential equation for v is simply

$$\frac{dv}{dt} = 12.$$

(b) Let $r(t)$ be the radius (in cm) of the balloon at time t. Since $v = \frac{4\pi r^3}{3}$, we have

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt},$$

and from (a) we know $\frac{dv}{dt} = 12$. We have

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dv}{dt} = \frac{12}{4\pi r^2} = \frac{3}{\pi r^2},$$

so the differential equation for r is

$$\frac{dr}{dt} = \frac{3}{\pi r^2}.$$

(c) The differential equation in (a) looks a little easier than (b). We find

$$v(t) = 12t + C_1$$

where C_1 is an arbitrary constant (which, in this case, gives the initial volume of the balloon). Then, since $r = \left(\frac{3v}{4\pi}\right)^{1/3}$, we have

$$r(t) = \left(\frac{36t + 3C_1}{4\pi}\right)^{1/3} = \left(\frac{9t}{\pi} + C_2\right)^{1/3},$$

where $C_2 = \frac{3C_1}{4\pi}$.