Math 308 - Differential Equations **Fall 2003**

Homework Assignment 2

Due *Wednesday, October 29.*

1. **Using Maple for Second Order Equations.** Work through the examples shown in the handout "Solving Second Order Differential Equations", and then use Maple to solve the following problems. (You should review the earlier Maple handouts, too.)

   (a) Consider the differential equation

   $$y'' + 3y' + y = t^2 e^{-t} \sin(t).$$

   i. (By hand, not with Maple.) Write down the correct form of the guess that you would use to find a particular solution to this problem by using the method of undetermined coefficients. *Do not solve for the coefficients!* Just write down the final guess–with all its undetermined coefficients–that is sure to work. Remember that this means you must find y_h to determine if your guess will work.

   ii. Use the `dsolve` command in Maple to find the general solution. Look at the solution carefully, and indicate which terms in the solution belong to y_h and which belong to Y_p. (You can print the Maple session and make notes on the print-out that you hand in.)

   (b) Use Maple to solve the initial value problem

   $$y'' + y' + 8y = te^{-2t} \cos(t), \quad y(0) = 1, \quad y'(0) = -1,$$

   and plot $y(t)$ for $-1 < t < 8$.

2. **A Deflection Anemometer.** Consider a spring-mass system as shown in Figure 3.8.10 on page 200 of the text (but, for the moment, we use τ for time and y for the displacement). The mass is m, the spring constant is k, the damping coefficient is γ, and $y(\tau)$ is the displacement of the mass at time τ. Assume that wind blows horizontally, parallel to the direction of motion of the mass, and that the force exerted on the mass by the wind is proportional to the wind speed $v(\tau)$. Then Newton's Second Law gives

   $$my''(\tau) + \gamma y'(\tau) + ky(\tau) = cv(\tau), \tag{1}$$

   where c is the proportionality constant that relates the wind speed to the force.

   (a) Show that by changing to the new function u defined by

   $$y(\tau) = \frac{c}{k} u(\omega_0 \tau), \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}},$$

   and then changing the independent variable from τ to a new variable $t = \omega_0 \tau$, we can rewrite the differential equation (1) as

   $$u''(t) + bu'(t) + u(t) = w(t) \tag{2}$$

   where

   $$b = \frac{\gamma}{\sqrt{mk}}, \quad \text{and} \quad w(t) = v(t/\omega_0) = v(\tau).$$

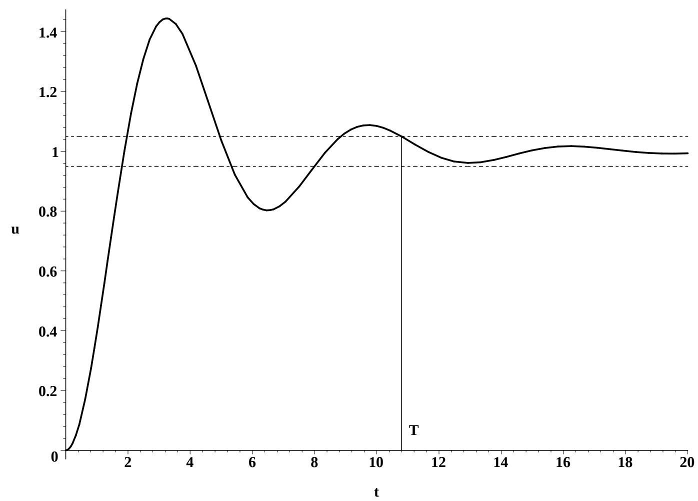
   (By choosing our coordinates in this way, we see that the problem is greatly simplified; there is really just one parameter that is important, not four.) We will use (2) for the rest of this problem. *Hint.* If $y(\tau) = \frac{c}{k} u(\omega_0 \tau)$, then

   $$\frac{dy}{d\tau} = \frac{d}{d\tau}\left(\frac{c}{k} u(\omega_0 \tau)\right) = \frac{c\omega_0}{k} u'(\omega_0 \tau) = \frac{c}{\sqrt{mk}} u'(t).$$

(b) Find $u_h(t)$, the solution to the homogeneous equation associated with (2). To give your answer, you will have to find b_c, the value of b for which the homogeneous system is critically damped. Then there are three cases to consider: $b < b_c$, $b = b_c$ and $b > b_c$.

(c) Suppose that for $t < 0$, there is no wind, and the mass is at rest. At $t = 0$, the wind speed $w(t)$ suddenly jumps to 1 and remains there. Solve the corresponding initial value problem for $u(t)$ for three cases: $b = b_c/3$, $b = b_c$ and $b = 3b_c$ (where b_c is the value found in the previous part). Plot all three solutions on one set of axes for $0 < t < 20$. Label each curve with the appropriate value of b. What is the steady-state behavior of $u(t)$ in each case?

(d) It should be clear from your plots in the previous part that if an anemometer displays $u(t)$ as an approximation to the wind speed $w(t)$, there will be a large error at first, but the error will approach zero asymptotically. Let's suppose that an error of 5% is "acceptable". The goal of this part of the problem is to design the anemometer so that the duration of the transient error is as short as possible. Since there is only one parameter to consider, *design* means find an appropriate value of b. Define T to be the time t at which $u(t)$ is within 5% of the correct steady-state value *for all* $t > T$. The plots of the previous part should make clear that T depends on the parameter b. The following plot shows an example of T in an underdamped case. Note that $u(t)$ crosses the "acceptable error" region several times, but it is only after $t = T \approx 10.8$ that the error remains within 5%. (The horizontal lines are $u = 0.95$ and $u = 1.05$.)



Find the value of b (to three significant digits[1]) that gives the smallest value of T for the initial value problem above. You can do this analytically, or by trial and error. Plot the solution for $0 < t < 20$.

[1] Remember that numbers such as 7.89 and 1.23×10^{-2} have three significant digits, but 0.003 has only one significant digit.

3. **An Economics Model.** We consider the relationship between the price of some commodity and the supply of that commodity. When the supply is too high, the price will decrease, and when the supply to too low, the price will increase. The simplest mathematical model of this is

$$\frac{dP}{dt} = -k_1(S - S_0),$$

where $P(t)$ is the price at time t, $S(t)$ is the supply, S_0 is the equilibrium supply at which the price does not change, and $k_1 > 0$ is a constant. We can include other influences on the price (such as inflation) by adding a function $f(t)$ to the right side:

$$\frac{dP}{dt} = -k_1(S - S_0) + f(t). \tag{3}$$

For example, $f(t) = \alpha$ models a constant inflation rate.

The supply also changes over time. When the price is high, the supplier increase production and S increases, while if the price is low, production (and there S) decrease. The simplest model of this is

$$\frac{dS}{dt} = k_2(P - P_0), \tag{4}$$

where $k_2 > 0$ and P_0 are constants. If $P > P_0$, the supply increases, while if $P < P_0$, the supply decreases.

(a) Find a second order differential equation for $P(t)$ by differentiating both sides of (3) and using (4) to eliminate S from the result.

(b) Suppose $f(t) = \alpha$, where α is a constant. Solve the differential equation in (a) for $P(t)$, and find the corresponding function $S(t)$. Describe the long term behavior of $P(t)$ and $S(t)$, and how the two functions are related. (Your solution will depend on the parameters P_0, S_0, k_1, k_2, and α.) Also discuss whether or not the long-term behavior depends on the initial conditions.

(c) Suppose we modify (4) so that it is

$$\frac{dS}{dt} = k_1(P - P_0) + r\frac{dP}{dt}, \tag{5}$$

where $r > 0$. Give an economic interpretation of the term $r\frac{dP}{dt}$ in this equation. Hint: The original equation says that whoever or whatever (production managers? the "invisble hand"?) decides the rate of change of S, it does so by considering only the difference between the current price and the equilibrium price P_0. Suppose it could also detect the current rate of change of the price; does the modified equation give a reasonable formula for how the rate of change of P should affect dS/dt?

(d) Repeat parts (a) and (b), but use the modified equation (5) given in (c) for the equation that governs S. That is, use (3) and (5) to find a second order differential equation for S, solve it, and describe the long-term behavior of $P(t)$ and $S(t)$. Does the long-term behavior depend on the initial conditions?