Homework Assignment 3

Due *Friday, November 7.*

For Questions 1-5, find the real-valued general solution, and then solve the given initial value problem.

1. $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ -6 & -4 \end{bmatrix} \vec{\mathbf{x}}, \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2. $\vec{\mathbf{x}}' = \begin{bmatrix} -4 & -1 \\ -2 & -3 \end{bmatrix} \vec{\mathbf{x}}, \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\vec{\mathbf{x}}' = \begin{bmatrix} 2/3 & 4 \\ -4 & 2/3 \end{bmatrix} \vec{\mathbf{x}}, \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

4. $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \vec{\mathbf{x}}, \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

5. $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \vec{\mathbf{x}}, \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

6. Suppose that for some matrix A, $A\vec{v}_1 = \lambda_1 \vec{v}_1$, and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, and $\lambda_1 \neq \lambda_2$. Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. (In other words, eigenvectors associated with *different* eigenvalues are always linearly independent.)

   Hint: Consider the equation
   $$k_1 \vec{v}_1 + k_2 \vec{v}_2 = \vec{0}. \tag{1}$$
   To show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, we must show that the only solution to this equation is $k_1 = k_2 = 0$. As a start, consider multiplying both sides of (1) by A to get a new equation. Then consider multiplying (1) by, say, λ_1 to get another new equation. Now subtract one of the new equations from the other new equation, and see where that leads.

7. Suppose $\vec{\mathbf{x}}_1(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{\mathbf{x}}_2(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ are solutions to the linear system
   $$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}, \quad \text{where} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
   Define the *Wronskian* of $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ to be
   $$W(t) = x(t)v(t) - u(t)y(t).$$
   (It is the determinant of the matrix whose columns are $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$.)

   (a) Find dW/dt.

   (b) Show that
   $$\frac{dW}{dt} = (a+d)W(t).$$
   Hint: use the fact that $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ solve the linear system.

(c) Find the solution to the first order differential equation $dW/dt = (a + d)W(t)$.

(d) Use (c) to show that if the vectors $\vec{\mathbf{x}}_1(0)$ and $\vec{\mathbf{x}}_2(0)$ are linearly independent, then $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are linearly independent for all t.

8. Let
$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$
Show that a is the only eigenvalue of A, and that *every* nonzero vector is an eigenvector.

9. Let
$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$
Find the eigenvalues and eigenvector of A. (This is a useful result to remember.)

10. Let
$$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}.$$
Find the eigenvalues and eigenvector of A. (Compare this to the previous problem.)