Homework Assignment 3 Solutions (Problems 8, 9, and 10)

8. We have $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, so $A - rI = \begin{bmatrix} a-r & 0 \\ 0 & a-r \end{bmatrix}$ and the characteristic polynomial is $\det(A - rI) = (a-r)^2$. Thus the only eigenvalue is $r = a$. Then

$$A - rI = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and therefore $(A - rI)\vec{v} = \vec{0}$ for *any* vector $\vec{v}$. So any nonzero vector $\vec{v}$ is an eigenvector.

We could also observe that for any vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

$$A\vec{v} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix} = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = a\vec{v}.$$

Thus, if $\vec{v} \neq \vec{0}$, $\vec{v}$ is an eigenvector associated with the eigenvalue a.

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9. We have $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. We'll have to assume that $a \neq d$, otherwise we have a repeated eigenvalue, which we haven't covered yet. Then $A - rI = \begin{bmatrix} a-r & b \\ 0 & d-r \end{bmatrix}$ and the characteristic polynomial is $\det(A - rI) = (a-r)(d-r)$. Thus the eigenvalues are

$$r_1 = a \quad \text{and} \quad r_2 = d.$$

Now find the eigenvectors.

For $\underline{r_1 = a}$, we have $A - r_1 I = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}$, so we may take $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\underline{r_2 = d}$, we have $A - r_2 I = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$, so we may take $\vec{v}_2 = \begin{bmatrix} b \\ d-a \end{bmatrix}$.

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10. We have $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, so $A - rI = \begin{bmatrix} a-r & b \\ c & -r \end{bmatrix}$ and the characteristic polynomial is $\det(A - rI) = (a-r)(-r) - bc = r^2 - ar - bc$. Thus the eigenvalues are

$$r = \frac{a \pm \sqrt{a^2 + 4bc}}{2}.$$

We now find the eigenvectors:

For $r_1 = \dfrac{a - \sqrt{a^2 + 4bc}}{2}$, we have $A - r_1 I = \begin{bmatrix} \frac{a+\sqrt{a^2+4bc}}{2} & b \\ c & \frac{-a+\sqrt{a^2+4bc}}{2} \end{bmatrix}$, so an eigenvector is $\vec{v}_1 = \begin{bmatrix} -b \\ \frac{a+\sqrt{a^2+4bc}}{2} \end{bmatrix}$.

For $r_2 = \dfrac{a + \sqrt{a^2 + 4bc}}{2}$, we have $A - r_2 I = \begin{bmatrix} \frac{a-\sqrt{a^2+4bc}}{2} & b \\ c & \frac{-a-\sqrt{a^2+4bc}}{2} \end{bmatrix}$, so an eigenvector is $\vec{v}_2 = \begin{bmatrix} -b \\ \frac{a-\sqrt{a^2+4bc}}{2} \end{bmatrix}$.

The main point to observe is that, unlike the upper triangular matrix in Problem 9, we can't simply read off the eigenvalues from entries of the matrix. The formulas for the eigenvalues and eigenvectors in this case are a bit messier.