Math 308 - Differential Equations **Fall 2003**

Homework Assignment 4

Due *Wednesday, December 3.*

For Questions 1-5,

(a) Find the general solution.

(b) Sketch the phase portrait:

- Where appropriate, include the straight line solutions (with arrows) corresponding to real eigenvalues and eigenvectors. Label these with the appropriate eigenvalue.
- Find and sketch the nullclines. Clearly label these as nullclines, so they are not confused with actual solutions. (Use a dashed line, or a different color.)
- Indicate the direction field on the nullclines and on the x and y axes.
- Sketch several trajectories to indicate the qualitative behavior of the solutions in the phase plane. Be sure your trajectories cross the axes and the nullclines with the correct angles, and if your trajectories approach the origin (as either $t \to \infty$ or $t \to -\infty$), be sure they approach it along the correct direction.

You may use Maple or PPLANE to check your answer.

(c) Classify the critical point as either a *nodal sink*, a *nodal source*, a *proper node*, an *improper node*, a *saddle*, a *spiral sink*, a *spiral source*, or a *center*; **and** state whether the equilibrium solution is *asymptotically stable*, *stable* (but not asymptotically stable), or *unstable*.

(d) If $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, sketch $x(t)$ and $y(t)$ as functions of t. (Assuming $\vec{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.)

1. $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \vec{\mathbf{x}}$

2. $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & 0 \\ -6 & -3 \end{bmatrix} \vec{\mathbf{x}}$

3. $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}$

4. $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix} \vec{\mathbf{x}}$

5. $\vec{\mathbf{x}}' = \begin{bmatrix} -2 & 3 \\ -3 & 2 \end{bmatrix} \vec{\mathbf{x}}$,

Problems 6 and 7 are on the back.

6. Consider the second order differential equation

$$\frac{d^2y}{dt^2} = 0.$$

(a) Find the general solution using basic calculus (i.e. just integrate).

(b) Find the general solution by following the same steps that one follows when solving the general second order homogeneous equation $ay'' + by' + cy = 0$. (In this case, $a = 1$, $b = 0$ and $c = 0$.)

(c) Convert the second order equation into a first order system of the form

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$$

by defining $v(t) = y'(t)$, and letting $\vec{\mathbf{x}}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$.

(d) Find the general solution of the system in (c).

(e) Sketch the phase portrait of the system in (c).

7. Consider the nonlinear system

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(3 - 2x - y)$$

(a) Find all the critical points. (Hint: There are four. Check your answers before proceeding.)

(b) Find the linearization at each critical point, classify the critical point (type and stability), and sketch the phase portrait for the linearized system.

(Hint: If you happen to have already figured this out in a previous part of this homework, you may refer to the previous work. You do not have to repeat it all here.)

(c) Sketch the phase portrait for the nonlinear system:

- Use the results of (b) to determine the phase portrait near each critical point.
- Use nullclines (of the *nonlinear system*) to help determine what happens elsewhere.
- Sketch and label the *separatrices* associated with any saddle points.

You may use Maple or PPLANE to check your answer.

(d) Consider the solution $(x(t), y(t))$ for which $x(0) = 1/2$ and $y(0) = 1/2$. Use your phase portrait in (c) to determine:

$$\lim_{t\to\infty} x(t), \quad \lim_{t\to-\infty} x(t), \quad \lim_{t\to\infty} y(t), \quad \lim_{t\to-\infty} y(t).$$