Homework Assignment 1

Due *Friday, January 28.*

1. Consider the first order differential equation

$$\frac{dy}{dt} = y(y+1)(y-2)^2$$

   (a) Find the equilibrium solutions.

   (b) In one set of axes, sketch the equilibrium solutions, and sketch several more solutions that indicate all the possible nonequilibrium behaviors of the system. (Include negative values of y and negative values of t in your plot.)

   (c) Suppose $y(0) = 1$. Find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.

2. In this problem, we consider a variation of a population model governed by the logistic equation. Suppose that a population grows according to the logistic equation, and in addition, part of the population is removed at a constant rate. This might happen if a population is hunted by humans, but only a maximum number of animals are allowed to be killed. Let $\alpha \geq 0$ be the rate at which part of the population is removed. The differential equation for the population is then

$$\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right)p - \alpha.$$

   (a) Find the equilibrium solutions. Note that the number of equilibria will depend on α. You should clearly indicate this in your answer.

   (b) Explain the consequences of including α in the system. Address the following issues:

      What happens if α is "too big"?
      Can the population be hunted to extinction?
      Is there a safe range of α for which the population does not become extinct?

   Illustrate your explanation by showing plots of the right side of the differential equation, and corresponding sketches of solutions, for several values of α.

3. Consider a variation of the Solow Growth Model in which L is governed by a logistic equation

$$\frac{dL}{dt} = \lambda\left(1 - \frac{L}{M}\right)L,$$

   where M is the carrying capacity (since K means capital in the Solow model), and λ is the small population growth rate.

   (a) Use the same procedure as in the lecture to find a first order differential equation for k, where $k = K/L$.

   (b) Explain why the resulting differential equation is nonautonomous. (A very short "explanation" is all that is needed.)

   (c) Predict what this model says about k as $t \to \infty$.

4. For each of the following equations, determine if it is separable. If it is, try to solve the initial value problem $y(0) = 1$. If you can not solve the initial value problem, state why.

(a) $\dfrac{dy}{dt} = y^2$

(b) $\dfrac{dy}{dt} = \dfrac{t}{y+1}$

(c) $\dfrac{dy}{dt} = y^2 + t$

(d) $\dfrac{dy}{dt} = \sin(y^2 + y)t$